Final Report
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Title: **Automated hyper-liking in an electronic mathematical proof-check journal**

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The main objective of the grant is to establish a hyper-linked electronic proof-check journal. The name of the journal is *Journal of Formalized Mathematics* (JFM). It is available at `http://mizar.uw.bialystok.pl/JFM`.

JFM consists of articles written originally in Mizar and translated mechanically into English. The original articles written in Mizar are in a machine readable form (and they are mechanically processed at the semantic level). This facilitates automatic insertion of hyper-links according to a uniform procedure. Automatic hyper-linking is fast: inserting more than 200 thousands hyper-links took about one hour, most of the time used for communication with the data base.

The book *A Compendium of Continuous Lattices* by G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott has been partially formalized in Mizar. The articles describing the formalization have been submitted and published in JFM.

The presented documentation consists of

- typical articles submitted to JFM describing the formalization of fragments of the compendium,
- the annual report of the form requested by CS Dept. of ONR.

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Galois Connections [1]

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Summary. The paper is the Mizar encoding of the Chapter 0 Section 3 of [12] In the paper the following concept are defined: Galois connections, Heyting algebras, and Boolean algebras.

MML Identifier: WAYBEL_1.
URL Address: http://mizar.uw.bialystok.pl/JFM/Vol8/waybel_1.html.

The articles [19], [21], [10], [22], [23], [8], [9], [17], [11], [7], [6], [20], [15], [18], [4], [2], [16], [5], [13], [1], [14], [3], and [24] provide the notation and terminology for this paper.

1. Preliminaries

Let A, B be non empty sets. Observe that every function from A into B is non empty.

Let L_1, L_2 be non empty 1-sorted structures and let f be a map from L_1 into L_2. Let us observe that f is one-to-one if and only if:

(Def.1) For all elements x, y of L_1 such that $f(x) = f(y)$ holds $x = y$.

One can prove the following proposition

(1) Let L be a non empty 1-sorted structure and let f be a map from L into L. If for every element x of L holds $f(x) = x$, then $f = \mathrm{id}_L$.

Let L_1, L_2 be non empty relation structures and let f be a map from L_1 into L_2. Let us observe that f is monotone if and only if:

(Def.2) For all elements x, y of L_1 such that $x \leq y$ holds $f(x) \leq f(y)$.

We now state four propositions:

(2) Let L be a non empty antisymmetric transitive relation structure with g.l.b.'s and let x, y, z be elements of L. If $x \leq y$, then $x \sqcap z \leq y \sqcap z$.

(3) Let L be a non empty antisymmetric transitive relation structure with l.u.b.'s and let x, y, z be elements of L. If $x \leq y$, then $x \sqcup z \leq y \sqcup z$.

(4) Let L be a non empty lower-bounded antisymmetric relation structure and let x be an element of L. Then if L has g.l.b.'s, then $\bot_L \sqcap x = \bot_L$ and if L is reflexive and transitive and has l.u.b.'s, then $\bot_L \sqcup x = x$.

(5) Let L be a non empty upper-bounded antisymmetric relation structure and let x be an element of L. Then if L is transitive and reflexive and has g.l.b.'s, then $\top_L \sqcap x = x$ and if L has l.u.b.'s, then $\top_L \sqcup x = \top_L$.

Let L be a non empty relation structure. We say that L is distributive if and only if:

[1]This work was partially supported by Office of Naval Research Grant N00014-95-1-1336.

(Def.3) For all elements x, y, z of L holds $x \sqcap (y \sqcup z) = x \sqcap y \sqcup x \sqcap z$.

We now state the proposition

(6) For every lattice L holds L is distributive iff for all elements x, y, z of L holds $x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$.

Let X be a set. Note that $2^X_\subseteq$ is distributive.

Let S be a non empty relation structure and let X be a set. We say that min X exists in S if and only if:

(Def.4) Inf X exists in S and $\bigsqcap_S X \in X$.

We introduce X has the minimum in S as a synonym of min X exists in S We say that max X exists in S if and only if:

(Def.5) Sup X exists in S and $\bigsqcup_S X \in X$.

We introduce X has the maximum in S as a synonym of max X exists in S.

Let S be a non empty relation structure, let s be an element of S, and let X be a set. The predicate $s = \min X$ is defined by:

(Def.6) Inf X exists in S and $s = \bigsqcap_S X$ and $\bigsqcap_S X \in X$.

The predicate $s = \max X$ is defined as follows:

(Def.7) Sup X exists in S and $s = \bigsqcup_S X$ and $\bigsqcup_S X \in X$.

Let L be a relation structure. One can verify that id_L is isomorphic.

Let L_1, L_2 be relation structures. We say that L_1 and L_2 are isomorphic if and only if:

(Def.8) There exists map from L_1 into L_2 which is isomorphic.

Let us observe that this predicate is reflexive.

The following two propositions are true:

(7) For all non empty relation structures L_1, L_2 such that L_1 and L_2 are isomorphic holds L_2 and L_1 are isomorphic.

(8) Let L_1, L_2, L_3 be relation structures. Suppose L_1 and L_2 are isomorphic and L_2 and L_3 are isomorphic. Then L_1 and L_3 are isomorphic.

2. Galois Connections

Let S, T be relation structures. A set is said to be a connection between S and T if:

(Def.9) There exists a map g from S into T and there exists a map d from T into S such that it = $\langle g, d\rangle$.

Let S, T be relation structures, let g be a map from S into T, and let d be a map from T into S. Then $\langle g, d\rangle$ is a connection between S and T.

Let S, T be non empty relation structures and let g_1 be a connection between S and T. We say that g_1 is Galois if and only if the condition (Def.10) is satisfied.

(Def.10) There exists a map g from S into T and there exists a map d from T into S such that

(i) $g_1 = \langle g, d\rangle$,

(ii) g is monotone,

(iii) d is monotone, and

(iv) for every element t of T and for every element s of S holds $t \leq g(s)$ iff $d(t) \leq s$.

The following proposition is true

(9) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. Then $\langle g, d\rangle$ is Galois if and only if the following conditions are satisfied:

(i) g is monotone,

(ii) d is monotone, and

(iii) for every element t of T and for every element s of S holds $t \leq g(s)$ iff $d(t) \leq s$.

Let S, T be non empty relation structures and let g be a map from S into T. We say that g is upper adjoint if and only if:

(Def.11) There exists a map d from T into S such that $\langle g, d \rangle$ is Galois.

We introduce g is lower-bounded as a synonym of g is upper adjoint.

Let S, T be non empty relation structures and let d be a map from T into S. We say that d is lower adjoint if and only if:

(Def.12) There exists a map g from S into T such that $\langle g, d \rangle$ is Galois.

We introduce d is upper-bounded as a synonym of d is lower adjoint.

One can prove the following propositions:

(10) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. If $\langle g, d \rangle$ is Galois, then g is upper adjoint and d is lower adjoint.

(11) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. Then $\langle g, d \rangle$ is Galois if and only if the following conditions are satisfied:
  (i) g is monotone, and
  (ii) for every element t of T holds $d(t) = \min g^{-1} \uparrow t$.

(12) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. Then $\langle g, d \rangle$ is Galois if and only if the following conditions are satisfied:
  (i) d is monotone, and
  (ii) for every element s of S holds $g(s) = \max d^{-1} \downarrow s$.

(13) Let S, T be non empty poset and let g be a map from S into T. If g is upper adjoint, then g is infs-preserving.

Let S, T be non empty poset. One can verify that every map from S into T which is upper adjoint is also infs-preserving.

Next we state the proposition

(14) Let S, T be non empty poset and let d be a map from T into S. If d is lower adjoint, then d is sups-preserving.

Let S, T be non empty poset. Observe that every map from S into T which is lower adjoint is also sups-preserving.

We now state a number of propositions:

(15) Let S, T be non empty poset and let g be a map from S into T. Suppose S is complete and g is infs-preserving. Then there exists a map d from T into S such that $\langle g, d \rangle$ is Galois and for every element t of T holds $d(t) = \min g^{-1} \uparrow t$.

(16) Let S, T be non empty poset and let d be a map from T into S. Suppose T is complete and d is sups-preserving. Then there exists a map g from S into T such that $\langle g, d \rangle$ is Galois and for every element s of S holds $g(s) = \max d^{-1} \downarrow s$.

(17) Let S, T be non empty poset and let g be a map from S into T. Suppose S is complete. Then g is infs-preserving if and only if g is monotone and g has a lower adjoint.

(18) Let S, T be non empty poset and let d be a map from T into S. Suppose T is complete. Then d is sups-preserving if and only if d is monotone and d has an upper adjoint.

(19) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. If $\langle g, d \rangle$ is Galois, then $d \cdot g \leq \mathrm{id}_S$ and $\mathrm{id}_T \leq g \cdot d$.

(20) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. Suppose g is monotone and d is monotone and $d \cdot g \leq \mathrm{id}_S$ and $\mathrm{id}_T \leq g \cdot d$. Then $\langle g, d \rangle$ is Galois.

(21) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. Suppose g is monotone and d is monotone and $d \cdot g \leq \mathrm{id}_S$ and $\mathrm{id}_T \leq g \cdot d$. Then $d = d \cdot g \cdot d$ and $g = g \cdot d \cdot g$.

(22) Let S, T be non empty relation structures, and let g be a map from S into T, and let d be a map from T into S. If $d = d \cdot g \cdot d$ and $g = g \cdot d \cdot g$, then $g \cdot d$ is idempotent and $d \cdot g$ is idempotent.

(23) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. Suppose $\langle g, d \rangle$ is Galois and g is onto. Let t be an element of T. Then $d(t) = \min g^{-1} \{t\}$.

(24) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. If for every element t of T holds $d(t) = \min g^{-1} \{t\}$, then $g \cdot d = \mathrm{id}_T$.

(25) Let L_1, L_2 be non empty 1-sorted structures, and let g_3 be a map from L_1 into L_2, and let g_2 be a map from L_2 into L_1. If $g_2 \cdot g_3 = \mathrm{id}_{(L_1)}$, then g_3 is one-to-one and g_2 is onto.

(26) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. If $\langle g, d \rangle$ is Galois, then g is onto iff d is one-to-one.

(27) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. Suppose $\langle g, d \rangle$ is Galois and d is onto. Let s be an element of S. Then $g(s) = \max d^{-1} \{s\}$.

(28) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. If for every element s of S holds $g(s) = \max d^{-1} \{s\}$, then $d \cdot g = \mathrm{id}_S$.

(29) Let S, T be non empty poset, and let g be a map from S into T, and let d be a map from T into S. If $\langle g, d \rangle$ is Galois, then g is one-to-one iff d is onto.

Let L be a non empty relation structure and let p be a map from L into L. We say that p is projection if and only if:

(Def.13) p is idempotent and monotone.

We introduce p is bounded as a synonym of p is projection.

Let L be a non empty relation structure. One can verify that id_L is projection.

Let L be a non empty relation structure. One can verify that there exists a map from L into L which is projection.

Let L be a non empty relation structure and let c be a map from L into L. We say that c is closure if and only if:

(Def.14) c is projection and $\mathrm{id}_L \leq c$.

We introduce c is join-inheriting as a synonym of c is closure.

Let L be a non empty relation structure. Observe that every map from L into L which is closure is also projection.

Let L be a non empty reflexive relation structure. Note that there exists a map from L into L which is closure.

Let L be a non empty reflexive relation structure. Observe that id_L is closure.

Let L be a non empty relation structure and let k be a map from L into L. We say that k is kernel if and only if:

(Def.15) k is projection and $k \leq \mathrm{id}_L$.

We introduce k is meet-inheriting as a synonym of k is kernel.

Let L be a non empty relation structure. Observe that every map from L into L which is kernel is also projection.

Let L be a non empty reflexive relation structure. Note that there exists a map from L into L which is kernel.

Let L be a non empty reflexive relation structure. Note that id_L is kernel.

The following two propositions are true:

(30) Let L be a non empty poset, and let c be a map from L into L, and let X be a subset of L. Suppose c is a closure operator and $\inf X$ exists in L and $X \subseteq \operatorname{rng} c$. Then $\inf X = c(\inf X)$.

(31) Let L be a non empty poset, and let k be a map from L into L, and let X be a subset of L. Suppose k is a kernel operator and $\sup X$ exists in L and $X \subseteq \operatorname{rng} k$. Then $\sup X = k(\sup X)$.

Let L_1, L_2 be non empty relation structures and let g be a map from L_1 into L_2. The functor g° yielding a map from L_1 into $\operatorname{Im} g$ is defined as follows:

(Def.16) $g^{\circ} = (\text{the carrier of } \operatorname{Im} g) \restriction (g)$.

The following proposition is true

(32) For all non empty relation structures L_1, L_2 and for every map g from L_1 into L_2 holds $g^{\circ} = g$.

Let L_1, L_2 be non empty relation structures and let g be a map from L_1 into L_2. Note that g° is onto.

One can prove the following proposition

(33) Let L_1, L_2 be non empty relation structures and let g be a map from L_1 into L_2. If g is monotone, then g° is monotone.

Let L_1, L_2 be non empty relation structures and let g be a map from L_1 into L_2. The functor $g_\circ$ yields a map from $\operatorname{Im} g$ into L_2 and is defined by:

(Def.17) $g_\circ = \mathrm{id}_{\operatorname{Im} g}$.

We now state the proposition

(34) Let L_1, L_2 be non empty relation structures, and let g be a map from L_1 into L_2, and let s be an element of $\operatorname{Im} g$. Then $g_\circ(s) = s$.

Let L_1, L_2 be non empty relation structures and let g be a map from L_1 into L_2. Note that $g_\circ$ is one-to-one and monotone.

The following propositions are true:

(35) For every non empty relation structure L and for every map f from L into L holds $f_\circ \cdot f^\circ = f$.

(36) For every non empty poset L and for every map f from L into L such that f is idempotent holds $f^\circ \cdot f_\circ = \mathrm{id}_{\operatorname{Im} f}$.

(37) Let L be a non empty poset and let f be a map from L into L. Suppose f is a projection operator. Then there exists a non empty poset T and there exists a map q from L into T and there exists a map i from T into L such that q is monotone and onto and i is monotone and one-to-one and $f = i \cdot q$ and $\mathrm{id}_T = q \cdot i$.

(38) Let L be a non empty poset and let f be a map from L into L. Given a non empty poset T and a map q from L into T and a map i from T into L such that q is monotone and i is monotone and $f = i \cdot q$ and $\mathrm{id}_T = q \cdot i$. Then f is a projection operator.

(39) For every non empty poset L and for every map f from L into L such that f is a closure operator holds $\langle f_\circ, f^\circ \rangle$ is Galois.

(40) Let L be a non empty poset and let f be a map from L into L. Suppose f is a closure operator. Then there exists a non empty poset S and there exists a map g from S into L and there exists a map d from L into S such that $\langle g, d \rangle$ is Galois and $f = g \cdot d$.

(41) Let L be a non empty poset and let f be a map from L into L. Suppose that

(i) f is monotone, and

(ii) there exists a non empty poset S and there exists a map g from S into L and there exists a map d from L into S such that $\langle g, d \rangle$ is Galois and $f = g \cdot d$.

Then f is a closure operator.

(42) For every non empty poset L and for every map f from L into L such that f is a kernel operator holds $\langle f^\circ, f_\circ \rangle$ is Galois.

(43) Let L be a non empty poset and let f be a map from L into L. Suppose f is a kernel operator. Then there exists a non empty poset T and there exists a map g from L into T and there exists a map d from T into L such that $\langle g, d \rangle$ is Galois and $f = d \cdot g$.

(44) Let L be a non empty poset and let f be a map from L into L. Suppose that

(i) f is monotone, and

(ii) there exists a non empty poset T and there exists a map g from L into T and there exists a map d from T into L such that $\langle g, d \rangle$ is Galois and $f = d \cdot g$.

Then f is a kernel operator.

(45) Let L be a non empty poset and let p be a map from L into L. Suppose p is a projection operator. Then $\operatorname{rng} p = \{c : c$ ranges over elements of L, $c \leq p(c)\} \cap \{k : k$ ranges over elements of L, $p(k) \leq k\}$.

(46) Let L be a non empty poset and let p be a map from L into L. Suppose p is a projection operator. Then

(i) $\{c : c$ ranges over elements of L, $c \leq p(c)\}$ is a non empty subset of L, and
(ii) $\{k : k$ ranges over elements of L, $p(k) \leq k\}$ is a non empty subset of L.

(47) Let L be a non empty poset and let p be a map from L into L. Suppose p is a projection operator. Then $\mathrm{rng}(p \restriction \{c : c$ ranges over elements of L, $c \leq p(c)\}) = \mathrm{rng}\, p$ and $\mathrm{rng}(p \restriction \{k : k$ ranges over elements of L, $p(k) \leq k\}) = \mathrm{rng}\, p$.

(48) Let L be a non empty poset and let p be a map from L into L. Suppose p is a projection operator. Let L_4 be a non empty subset of L and let L_5 be a non empty subset of L. Suppose $L_4 = \{c : c$ ranges over elements of L, $c \leq p(c)\}$. Then $p \restriction L_4$ is a map from $\mathrm{sub}(L_4)$ into $\mathrm{sub}(L_4)$.

(49) Let L be a non empty poset and let p be a map from L into L. Suppose p is a projection operator. Let L_5 be a non empty subset of L. Suppose $L_5 = \{k : k$ ranges over elements of L, $p(k) \leq k\}$. Then $p \restriction L_5$ is a map from $\mathrm{sub}(L_5)$ into $\mathrm{sub}(L_5)$.

(50) Let L be a non empty poset and let p be a map from L into L. Suppose p is a projection operator. Let L_4 be a non empty subset of L. Suppose $L_4 = \{c : c$ ranges over elements of L, $c \leq p(c)\}$. Let p_1 be a map from $\mathrm{sub}(L_4)$ into $\mathrm{sub}(L_4)$. If $p_1 = p \restriction L_4$, then p_1 is a closure operator.

(51) Let L be a non empty poset and let p be a map from L into L. Suppose p is a projection operator. Let L_5 be a non empty subset of L. Suppose $L_5 = \{k : k$ ranges over elements of L, $p(k) \leq k\}$. Let p_2 be a map from $\mathrm{sub}(L_5)$ into $\mathrm{sub}(L_5)$. If $p_2 = p \restriction L_5$, then p_2 is a kernel operator.

(52) Let L be a non empty poset and let p be a map from L into L. Suppose p is monotone. Let L_4 be a subset of L. If $L_4 = \{c : c$ ranges over elements of L, $c \leq p(c)\}$, then $\mathrm{sub}(L_4)$ is sups-inheriting.

(53) Let L be a non empty poset and let p be a map from L into L. Suppose p is monotone. Let L_5 be a subset of L. If $L_5 = \{k : k$ ranges over elements of L, $p(k) \leq k\}$, then $\mathrm{sub}(L_5)$ is infs-inheriting.

(54) Let L be a non empty poset and let p be a map from L into L. Suppose p is a projection operator. Let L_4 be a non empty subset of L. Suppose $L_4 = \{c : c$ ranges over elements of L, $c \leq p(c)\}$. Then
(i) if p is infs-preserving, then $\mathrm{sub}(L_4)$ is infs-inheriting and $\mathrm{Im}\, p$ is infs-inheriting, and
(ii) if p is filtered-infs-preserving, then $\mathrm{sub}(L_4)$ is filtered-infs-inheriting and $\mathrm{Im}\, p$ is filtered-infs-inheriting.

(55) Let L be a non empty poset and let p be a map from L into L. Suppose p is a projection operator. Let L_5 be a non empty subset of L. Suppose $L_5 = \{k : k$ ranges over elements of L, $p(k) \leq k\}$. Then
(i) if p is sups-preserving, then $\mathrm{sub}(L_5)$ is sups-inheriting and $\mathrm{Im}\, p$ is sups-inheriting, and
(ii) if p is directed-sups-preserving, then $\mathrm{sub}(L_5)$ is directed-sups-inheriting and $\mathrm{Im}\, p$ is directed-sups-inheriting.

(56) Let L be a non empty poset and let p be a map from L into L. Then if p is a closure operator, then $\mathrm{Im}\, p$ is infs-inheriting and if p is a kernel operator, then $\mathrm{Im}\, p$ is sups-inheriting.

(57) Let L be a complete non empty poset and let p be a map from L into L. If p is a projection operator, then $\mathrm{Im}\, p$ is complete.

(58) Let L be a non empty poset and let c be a map from L into L. Suppose c is a closure operator. Then
(i) c° is sups-preserving, and
(ii) for every subset X of L such that $X \subseteq$ the carrier of $\mathrm{Im}\, c$ and sup X exists in L holds sup X exists in $\mathrm{Im}\, c$ and $\bigsqcup_{\mathrm{Im}\, c} X = c(\bigsqcup_L X)$.

(59) Let L be a non empty poset and let k be a map from L into L. Suppose k is a kernel operator. Then
(i) k° is infs-preserving, and

(ii) for every subset X of L such that $X \subseteq$ the carrier of $\operatorname{Im} k$ and inf X exists in L holds inf X exists in $\operatorname{Im} k$ and $\bigsqcap_{\operatorname{Im} k} X = k(\bigsqcap_L X)$.

3. Heyting Algebra

One can prove the following propositions:

(60) For every complete non empty poset L holds $\langle \operatorname{IdsMap}(L), \operatorname{SupMap}(L) \rangle$ is Galois and $\operatorname{SupMap}(L)$ is sups-preserving.

(61) For every complete non empty poset L holds $\operatorname{IdsMap}(L) \cdot \operatorname{SupMap}(L)$ is a closure operator and $\operatorname{Im}(\operatorname{IdsMap}(L) \cdot \operatorname{SupMap}(L))$ and L are isomorphic.

Let S be a non empty relation structure and let x be an element of S. The functor $x \sqcap -$ yielding a map from S into S is defined by:

(Def.18) For every element s of S holds $(x \sqcap -)(s) = x \sqcap s$.

The following propositions are true:

(62) For every non empty relation structure S and for all elements x, t of S holds $\{s : s$ ranges over elements of S, $x \sqcap s \leq t\} = (x \sqcap -)^{-1} {\downarrow} t$.

(63) For every non empty semilattice S and for every element x of S holds $x \sqcap -$ is monotone.

Let S be a non empty semilattice and let x be an element of S. Observe that $x \sqcap -$ is monotone.

Next we state several propositions:

(64) Let S be a non empty relation structure, and let x be an element of S, and let X be a subset of S. Then $(x \sqcap -)^{\circ} X = \{x \sqcap y : y$ ranges over elements of S, $y \in X\}$.

(65) Let S be a non empty semilattice. Then for every element x of S holds $x \sqcap -$ has an upper adjoint if and only if for all elements x, t of S holds max $\{s : s$ ranges over elements of S, $x \sqcap s \leq t\}$ exists in S.

(66) Let S be a non empty semilattice. Suppose that for every element x of S holds $x \sqcap -$ has an upper adjoint. Let X be a subset of S. Suppose sup X exists in S. Let x be an element of S. Then $x \sqcap \bigsqcup_S X = \bigsqcup_S \{x \sqcap y : y$ ranges over elements of S, $y \in X\}$.

(67) Let S be a complete non empty poset. Then for every element x of S holds $x \sqcap -$ has an upper adjoint if and only if for every subset X of S and for every element x of S holds $x \sqcap \bigsqcup_S X = \bigsqcup_S \{x \sqcap y : y$ ranges over elements of S, $y \in X\}$.

(68) Let S be a non empty lattice. Suppose that for every subset X of S such that sup X exists in S and for every element x of S holds $x \sqcap \bigsqcup_S X = \bigsqcup_S \{x \sqcap y : y$ ranges over elements of S, $y \in X\}$. Then S is distributive.

Let H be a non empty relation structure. We say that H is Heyting if and only if:

(Def.19) H is a lattice and for every element x of H holds $x \sqcap -$ has an upper adjoint.

We introduce H is sups-inheriting as a synonym of H is Heyting.

Let us note that every non empty relation structure which is Heyting is also reflexive transitive and antisymmetric and has g.l.b.'s and l.u.b.'s.

Let H be a non empty relation structure and let a be an element of H. Let us assume that H is Heyting. The functor $a \Rightarrow -$ yields a map from H into H and is defined as follows:

(Def.20) $\langle a \Rightarrow -, a \sqcap - \rangle$ is Galois.

One can prove the following proposition

(69) For every non empty relation structure H such that H is a Heyting algebra holds H is distributive.

One can verify that every non empty relation structure which is Heyting is also distributive.

Let H be a non empty relation structure and let a, y be elements of H. The functor $a \Rightarrow y$ yielding an element of H is defined by:

(Def.21) $a \Rightarrow y = (a \Rightarrow -)(y)$.

We now state two propositions:

(70) Let H be a non empty relation structure. Suppose H is a Heyting algebra. Let x, a, y be elements of H. Then $x \geq a \sqcap y$ if and only if $a \Rightarrow x \geq y$.

(71) For every non empty relation structure H such that H is a Heyting algebra holds H is upper-bounded.

Let us note that every non empty relation structure which is Heyting is also upper-bounded.

The following propositions are true:

(72) Let H be a non empty relation structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\top_H = a \Rightarrow b$ if and only if $a \leq b$.

(73) For every non empty relation structure H such that H is a Heyting algebra and for every element a of H holds $\top_H = a \Rightarrow a$.

(74) Let H be a non empty relation structure. Suppose H is a Heyting algebra. Let a, b be elements of H. If $\top_H = a \Rightarrow b$ and $\top_H = b \Rightarrow a$, then $a = b$.

(75) For every non empty relation structure H such that H is a Heyting algebra and for all elements a, b of H holds $b \leq a \Rightarrow b$.

(76) For every non empty relation structure H such that H is a Heyting algebra and for every element a of H holds $\top_H = a \Rightarrow \top_H$.

(77) For every non empty relation structure H such that H is a Heyting algebra and for every element b of H holds $b = \top_H \Rightarrow b$.

(78) Let H be a non empty relation structure. Suppose H is a Heyting algebra. Let a, b, c be elements of H. If $a \leq b$, then $b \Rightarrow c \leq a \Rightarrow c$.

(79) Let H be a non empty relation structure. Suppose H is a Heyting algebra. Let a, b, c be elements of H. If $b \leq c$, then $a \Rightarrow b \leq a \Rightarrow c$.

(80) Let H be a non empty relation structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $a \sqcap (a \Rightarrow b) = a \sqcap b$.

(81) Let H be a non empty relation structure. Suppose H is a Heyting algebra. Let a, b, c be elements of H. Then $a \sqcup b \Rightarrow c = (a \Rightarrow c) \sqcap (b \Rightarrow c)$.

Let H be a non empty relation structure and let a be an element of H. The functor $\neg a$ yielding an element of H is defined by:

(Def.22) $\neg a = a \Rightarrow \bot_H$.

One can prove the following propositions:

(82) Let H be a non empty relation structure. Suppose H is a Heyting algebra and lower-bounded. Let a be an element of H. Then $\neg a = \max\{x : x$ ranges over elements of H, $a \sqcap x = \bot_H\}$.

(83) Let H be a non empty relation structure. If H is a Heyting algebra and lower-bounded, then $\neg(\bot_H) = \top_H$ and $\neg(\top_H) = \bot_H$.

(84) Let H be a non empty lower-bounded relation structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\neg a \geq b$ if and only if $\neg b \geq a$.

(85) Let H be a non empty lower-bounded relation structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\neg a \geq b$ if and only if $a \sqcap b = \bot_H$.

(86) Let H be a non empty lower-bounded relation structure. Suppose H is a Heyting algebra. Let a, b be elements of H. If $a \leq b$, then $\neg b \leq \neg a$.

(87) Let H be a non empty lower-bounded relation structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\neg(a \sqcup b) = \neg a \sqcap \neg b$.

(88) Let H be a non empty lower-bounded relation structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\neg(a \sqcap b) \geq \neg a \sqcup \neg b$.

Let L be a non empty relation structure and let x, y be elements of L. We say that y is a complement of x if and only if:

(Def.23) $x \sqcup y = \top_L$ and $x \sqcap y = \bot_L$.

Let L be a non empty relation structure. We say that L is complemented if and only if:

(Def.24) For every element x of L holds there exists element of L which is a complement of x.

Let X be a set. One can verify that $2^X_\subseteq$ is complemented.

We now state two propositions:

(89) Let L be a non empty bounded lattice. Suppose L is a Heyting algebra and for every element x of L holds $\neg\neg x = x$. Let x be an element of L. Then $\neg x$ is a complement of x.

(90) Let L be a non empty bounded lattice. Then L is distributive and complemented if and only if L is a Heyting algebra and for every element x of L holds $\neg\neg x = x$.

Let B be a non empty relation structure. We say that B is Boolean if and only if:

(Def.25) B is a lattice bounded distributive and complemented.

We introduce B is relation structure yielding and B is filtered as synonyms of B is Boolean.

One can verify that every non empty relation structure which is Boolean is also reflexive transitive antisymmetric bounded distributive and complemented and has g.l.b.'s and l.u.b.'s.

One can check that every non empty relation structure which is reflexive transitive antisymmetric bounded distributive and complemented and has g.l.b.'s and l.u.b.'s is also Boolean.

Let us observe that every non empty relation structure which is Boolean is also Heyting.

Let us observe that there exists a lattice which is strict Boolean and non empty.

One can check that there exists a lattice which is strict Heyting and non empty.

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The "Way-Below" Relation [1]

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Summary. In the paper the "way-below" relation, in symbols $x \ll y$, is introduced. Some authors prefer the term "relatively compact" or "way inside", since in the poset of open sets of a topology it is natural to read $U \ll V$ as "U is relatively compact in V". A compact element of a poset (or an element isolated from below) is defined to be way below itself. So, compactness in the poset of open sets of a topology is equivalent to compactness in that topology.

The article includes definitions, facts and examples 1.1–1.8 presented in [15, pp. 38–42].

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The articles [5], [25], [29], [30], [31], [20], [14], [23], [8], [28], [10], [11], [22], [24], [6], [19], [7], [26], [33], [27], [21], [32], [13], [12], [9], [4], [2], [1], [16], [3], [17], and [18] provide the notation and terminology for this paper.

1. The "Way-Below" Relation

Let L be a non empty reflexive relation structure and let x, y be elements of L. We say that x is way below y if and only if:

(Def.1) For every non empty directed subset D of L such that $y \leq \sup D$ there exists an element d of L such that $d \in D$ and $x \leq d$.

We introduce $x \ll y$ and $y \gg x$ as synonyms of x is way below y.

Let L be a non empty reflexive relation structure and let x be an element of L. We say that x is compact if and only if:

(Def.2) x is way below x.

We introduce x is isolated from below as a synonym of x is compact.

One can prove the following propositions:

(1) Let L be a non empty reflexive antisymmetric relation structure and let x, y be elements of L. If $x \ll y$, then $x \leq y$.

(2) Let L be a non empty reflexive transitive relation structure and let u, x, y, z be elements of L. If $u \leq x$ and $x \ll y$ and $y \leq z$, then $u \ll z$.

(3) Let L be a non empty poset. Suppose L is inf-complete or has l.u.b.'s. Let x, y, z be elements of L. If $x \ll z$ and $y \ll z$, then sup $\{x, y\}$ exists in L and $x \sqcup y \ll z$.

(4) Let L be a lower-bounded antisymmetric reflexive non empty relation structure and let x be an element of L. Then $\perp_L \ll x$.

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(5) For every non empty poset L and for all elements x, y, z of L such that $x \ll y$ and $y \ll z$ holds $x \ll z$.

(6) Let L be a non empty reflexive antisymmetric relation structure and let x, y be elements of L. If $x \ll y$ and $x \gg y$, then $x = y$.

Let L be a non empty reflexive relation structure and let x be an element of L. The functor $\Downarrow x$ yielding a subset of L is defined by:

(Def.3) $\Downarrow x = \{y : y$ ranges over elements of L, $y \ll x\}$.

The functor $\Uparrow x$ yielding a subset of L is defined as follows:

(Def.4) $\Uparrow x = \{y : y$ ranges over elements of L, $y \gg x\}$.

The following propositions are true:

(7) For every non empty reflexive relation structure L and for all elements x, y of L holds $x \in \Downarrow y$ iff $x \ll y$.

(8) For every non empty reflexive relation structure L and for all elements x, y of L holds $x \in \Uparrow y$ iff $x \gg y$.

(9) For every non empty reflexive antisymmetric relation structure L and for every element x of L holds $x \geq \Downarrow x$.

(10) For every non empty reflexive antisymmetric relation structure L and for every element x of L holds $x \leq \Uparrow x$.

(11) For every non empty reflexive antisymmetric relation structure L and for every element x of L holds $\Downarrow x \subseteq \downarrow x$ and $\Uparrow x \subseteq \uparrow x$.

(12) Let L be a non empty reflexive transitive relation structure and let x, y be elements of L. If $x \leq y$, then $\Downarrow x \subseteq \Downarrow y$ and $\Uparrow y \subseteq \Uparrow x$.

Let L be a lower-bounded non empty reflexive antisymmetric relation structure and let x be an element of L. One can check that $\Downarrow x$ is non empty.

Let L be a non empty reflexive transitive relation structure and let x be an element of L. One can verify that $\Downarrow x$ is lower and $\Uparrow x$ is upper.

Let L be a sup-semilattice and let x be an element of L. One can check that $\Downarrow x$ is directed.

Let L be an inf-complete non empty poset and let x be an element of L. Observe that $\Downarrow x$ is directed.

Let L be a connected non empty relation structure. Note that every subset of L is directed and filtered.

Let us note that every non empty chain which is up-complete and lower-bounded is also complete.

Let us observe that there exists a non empty chain which is complete.

One can prove the following propositions:

(13) For every up-complete non empty chain L and for all elements x, y of L such that $x < y$ holds $x \ll y$.

(14) Let L be a non empty reflexive antisymmetric relation structure and let x, y be elements of L. If x is not compact and $x \ll y$, then $x < y$.

(15) For every non empty lower-bounded reflexive antisymmetric relation structure L holds $\bot_L$ is compact.

(16) For every up-complete non empty poset L and for every non empty finite directed subset D of L holds $\sup D \in D$.

(17) For every up-complete non empty poset L such that L is finite holds every element of L is isolated from below.

2. The Way-Below Relation in Other Terms

The scheme *SSubsetEx* deals with a non empty relation structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a subset X of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $x \in X$ iff $\mathcal{P}[x]$
for all values of the parameters.

The following propositions are true:

(18) Let L be a complete lattice and let x, y be elements of L. Suppose $x \ll y$. Let X be a subset of L. If $y \leq \sup X$, then there exists a finite subset A of L such that $A \subseteq X$ and $x \leq \sup A$.

(19) Let L be a complete lattice and let x, y be elements of L. Suppose that for every subset X of L such that $y \leq \sup X$ there exists a finite subset A of L such that $A \subseteq X$ and $x \leq \sup A$. Then $x \ll y$.

(20) Let L be a non empty reflexive transitive relation structure and let x, y be elements of L. If $x \ll y$, then for every ideal I of L such that $y \leq \sup I$ holds $x \in I$.

(21) Let L be an up-complete non empty poset and let x, y be elements of L. If for every ideal I of L such that $y \leq \sup I$ holds $x \in I$, then $x \ll y$.

(22) Let L be a lower-bounded lattice. Suppose L is meet-continuous. Let x, y be elements of L. Then $x \ll y$ if and only if for every ideal I of L such that $y = \sup I$ holds $x \in I$.

(23) Let L be a complete lattice. Then every element of L is compact if and only if for every non empty subset X of L there exists an element x of L such that $x \in X$ and for every element y of L such that $y \in X$ holds $x \not< y$.

3. Continuous Lattices

Let L be a non empty reflexive relation structure. We say that L satisfies axiom of approximation if and only if:

(Def.5) For every element x of L holds $x = \sup \downdownarrows x$.

Let us mention that every non empty reflexive relation structure which is trivial satisfies axiom of approximation.

Let L be a non empty reflexive relation structure. We say that L is continuous if and only if:

(Def.6) For every element x of L holds $\downdownarrows x$ is non empty and directed and L is up-complete and satisfies axiom of approximation.

Let us observe that every non empty reflexive relation structure which is continuous is also up-complete and satisfies axiom of approximation and every lower-bounded sup-semilattice which is up-complete and satisfies axiom of approximation is also continuous.

One can check that there exists a lattice which is continuous complete and strict.

Let L be a continuous non empty reflexive relation structure and let x be an element of L. One can check that $\downdownarrows x$ is non empty and directed.

We now state two propositions:

(24) Let L be an up-complete semilattice. Suppose that for every element x of L holds $\downdownarrows x$ is non empty and directed. Then L satisfies axiom of approximation if and only if for all elements x, y of L such that $x \not\leq y$ there exists an element u of L such that $u \ll x$ and $u \not\leq y$.

(25) For every continuous lattice L and for all elements x, y of L holds $x \leq y$ iff $\downdownarrows x \subseteq \downdownarrows y$.

One can verify that every non empty chain which is complete satisfies axiom of approximation.

The following proposition is true

(26) For every complete lattice L such that every element of L is compact holds L satisfies axiom of approximation.

4. The Way-Below Relation in Direct Powers

Let f be a binary relation. We say that f is nonempty if and only if:

(Def.7) For every 1-sorted structure S such that $S \in \operatorname{rng} f$ holds S is non empty.

We say that f is reflexive-yielding if and only if:

(Def.8) For every relation structure S such that $S \in \operatorname{rng} f$ holds S is reflexive.

Let I be a set. Note that there exists a many sorted set indexed by I which is relation structure yielding nonempty and reflexive-yielding.

Let I be a set and let J be a relation structure yielding nonempty many sorted set indexed by I. One can check that $\prod J$ is non empty.

Let I be a non empty set, let J be a relation structure yielding nonempty many sorted set indexed by I, and let i be an element of I. Then $J(i)$ is a non empty relation structure.

Let I be a set and let J be a relation structure yielding nonempty many sorted set indexed by I. One can verify that every element of $\prod J$ is function-like and relation-like.

Let I be a non empty set, let J be a relation structure yielding nonempty many sorted set indexed by I, let x be an element of $\prod J$, and let i be an element of I. Then $x(i)$ is an element of $J(i)$.

Let I be a non empty set, let J be a relation structure yielding nonempty many sorted set indexed by I, let i be an element of I, and let X be a subset of $\prod J$. Then $\pi_i X$ is a subset of $J(i)$.

Next we state two propositions:

(27) Let I be a non empty set, and let J be a relation structure yielding nonempty many sorted set indexed by I, and let x be a function. Then x is an element of $\prod J$ if and only if $\operatorname{dom} x = I$ and for every element i of I holds $x(i)$ is an element of $J(i)$.

(28) Let I be a non empty set, and let J be a relation structure yielding nonempty many sorted set indexed by I, and let x, y be elements of $\prod J$. Then $x \leq y$ if and only if for every element i of I holds $x(i) \leq y(i)$.

Let I be a non empty set and let J be a relation structure yielding nonempty reflexive-yielding many sorted set indexed by I. Note that $\prod J$ is reflexive. Let i be an element of I. Then $J(i)$ is a non empty reflexive relation structure.

Let I be a non empty set, let J be a relation structure yielding nonempty reflexive-yielding many sorted set indexed by I, let x be an element of $\prod J$, and let i be an element of I. Then $x(i)$ is an element of $J(i)$.

The following propositions are true:

(29) Let I be a non empty set and let J be a relation structure yielding nonempty many sorted set indexed by I. If for every element i of I holds $J(i)$ is transitive, then $\prod J$ is transitive.

(30) Let I be a non empty set and let J be a relation structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is antisymmetric. Then $\prod J$ is antisymmetric.

(31) Let I be a non empty set and let J be a relation structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a complete lattice. Then $\prod J$ is a complete lattice.

(32) Let I be a non empty set and let J be a relation structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a complete lattice. Let X be a subset of $\prod J$ and let i be an element of I. Then $(\sup X)(i) = \sup \pi_i X$.

(33) Let I be a non empty set and let J be a relation structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a complete lattice. Let x, y be elements of $\prod J$. Then $x \ll y$ if and only if the following conditions are satisfied:

(i) for every element i of I holds $x(i) \ll y(i)$, and

(ii) there exists a finite subset K of I such that for every element i of I such that $i \notin K$ holds $x(i) = \bot_{J(i)}$.

5. The Way-Below Relation in Topological Spaces

We now state four propositions:

(34) Let T be a non empty topological space and let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose x is way below y. Let F be a family of subsets of T. If F is open and $y \subseteq \bigcup F$, then there exists a finite subset G of F such that $x \subseteq \bigcup G$.

(35) Let T be a non empty topological space and let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose that for every family F of subsets of T such that F is open and $y \subseteq \bigcup F$ there exists a finite subset G of F such that $x \subseteq \bigcup G$. Then x is way below y.

(36) Let T be a non empty topological space, and let x be an element of $\langle$the topology of T, $\subseteq\rangle$, and let X be a subset of T. If $x = X$, then x is compact iff X is compact.

(37) Let T be a non empty topological space and let x be an element of $\langle$the topology of T, $\subseteq\rangle$. Suppose $x =$ the carrier of T. Then x is compact if and only if T is compact.

Let T be a non empty topological space. We say that T is locally-compact if and only if the condition (Def.9) is satisfied.

(Def.9) Let x be a point of T and let X be a subset of T. Suppose $x \in X$ and X is open. Then there exists a subset Y of T such that $x \in \operatorname{Int} Y$ and $Y \subseteq X$ and Y is compact.

One can check that every non empty topological space which is compact and T_2 is also T_3 T_4 and locally-compact.

The following proposition is true

(38) For every set x holds $\{x\}_{\rm top}$ is T_2.

Let us observe that there exists a non empty topological space which is compact and T_2 .

Next we state two propositions:

(39) Let T be a non empty topological space and let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. If there exists a subset Z of T such that $x \subseteq Z$ and $Z \subseteq y$ and Z is compact, then $x \ll y$.

(40) Let T be a non empty topological space. Suppose T is locally-compact. Let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. If $x \ll y$, then there exists a subset Z of T such that $x \subseteq Z$ and $Z \subseteq y$ and Z is compact.

Let T be a topological structure and let X be a subset of the carrier of T. Then $\overline{X}$ is a subset of T.

One can prove the following propositions:

(41) Let T be a non empty topological space. Suppose T is locally-compact and a T_2 space. Let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. If $x \ll y$, then there exists a subset Z of T such that $Z = x$ and $\overline{Z} \subseteq y$ and $\overline{Z}$ is compact.

(42) Let X be a non empty topological space. Suppose X is a T_3 space and $\langle$the topology of X, $\subseteq\rangle$ is continuous. Then X is locally-compact.

(43) For every non empty topological space T such that T is locally-compact holds $\langle$the topology of T, $\subseteq\rangle$ is continuous.

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Auxiliary and Approximating Relations [1]

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Summary. The goal of this paper is to formalize the second part of Chapter I Section 1 in [12]. Definitions of auxiliary and approximating relations are introduced in this work. Subsections 1.9–1.19 are formalized.

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The articles [22], [25], [19], [11], [23], [24], [20], [10], [2], [26], [28], [7], [8], [27], [6], [1], [21], [18], [4], [17], [13], [29], [14], [15], [3], [9], [16], and [5] provide the notation and terminology for this paper.

1. Auxiliary Relations

Let L be a 1-sorted structure.

(Def.1) A binary relation on the carrier of L is called a binary relation on L.

Let L be a non empty reflexive relation structure. The functor (L) −waybelow yielding a binary relation on L is defined by:

(Def.2) For all elements x, y of L holds $\langle x, y\rangle \in (L)$ −waybelow iff $x \ll y$.

Let L be a relation structure. The functor $\mathrm{IntRel}(L)$ yields a binary relation on L and is defined as follows:

(Def.3) $\mathrm{IntRel}(L) =$ the internal relation of L.

Let L be a relation structure and let R be a binary relation on L. We say that R is auxiliary(i) if and only if:

(Def.4) For all elements x, y of L such that $\langle x, y\rangle \in R$ holds $x \leq y$.

We say that R is auxiliary(ii) if and only if:

(Def.5) For all elements x, y, z, u of L such that $u \leq x$ and $\langle x, y\rangle \in R$ and $y \leq z$ holds $\langle u, z\rangle \in R$.

Let L be a non empty relation structure and let R be a binary relation on L. We say that R is auxiliary(iii) if and only if:

(Def.6) For all elements x, y, z of L such that $\langle x, z\rangle \in R$ and $\langle y, z\rangle \in R$ holds $\langle x \sqcup y, z\rangle \in R$.

We say that R is auxiliary(iv) if and only if:

(Def.7) For every element x of L holds $\langle \bot_L, x\rangle \in R$.

Let L be a non empty relation structure and let R be a binary relation on L. We say that R is auxiliary if and only if:

(Def.8) R is auxiliary(i) auxiliary(ii) auxiliary(iii) and auxiliary(iv).

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Let L be a non empty relation structure. Observe that every binary relation on L which is auxiliary is also auxiliary(i) auxiliary(ii) auxiliary(iii) and auxiliary(iv) and every binary relation on L which is auxiliary(i) auxiliary(ii) auxiliary(iii) and auxiliary(iv) is also auxiliary.

Let L be a lower-bounded transitive antisymmetric relation structure with l.u.b.'s. Observe that there exists a binary relation on L which is auxiliary.

One can prove the following proposition

(1) Let L be a lower-bounded sup-semilattice, and let A_1 be an auxiliary binary relation on L, and let x, y, z, u be elements of L. If $\langle x, z\rangle \in A_1$ and $\langle y, u\rangle \in A_1$, then $\langle x \sqcup y, z \sqcup u\rangle \in A_1$.

Let L be a lower-bounded sup-semilattice. Note that every binary relation on L which is auxiliary is also transitive.

Let L be a relation structure. Note that $\mathrm{IntRel}(L)$ is auxiliary(i).

Let L be a transitive relation structure. Observe that $\mathrm{IntRel}(L)$ is auxiliary(ii).

Let L be an antisymmetric relation structure with l.u.b.'s. Note that $\mathrm{IntRel}(L)$ is auxiliary(iii).

Let L be a lower-bounded antisymmetric non empty relation structure. Observe that $\mathrm{IntRel}(L)$ is auxiliary(iv).

In the sequel a is a set.

Let L be a lower-bounded sup-semilattice. The functor $\mathrm{Aux}(L)$ is defined by:

(Def.9) $a \in \mathrm{Aux}(L)$ iff a is an auxiliary binary relation on L.

Let L be a lower-bounded sup-semilattice. Note that $\mathrm{Aux}(L)$ is non empty.

One can prove the following two propositions:

(2) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $A_1 \subseteq \mathrm{IntRel}(L)$.

(3) For every lower-bounded sup-semilattice L holds $\top_{\langle \mathrm{Aux}(L), \subseteq\rangle} = \mathrm{IntRel}(L)$.

Let L be a lower-bounded sup-semilattice. One can verify that $\langle \mathrm{Aux}(L), \subseteq\rangle$ is upper-bounded.

Let L be a non empty relation structure. The functor $\mathrm{AuxBottom}(L)$ yielding a binary relation on L is defined by:

(Def.10) For all elements x, y of L holds $\langle x, y\rangle \in \mathrm{AuxBottom}(L)$ iff $x = \perp_L$.

Let L be a lower-bounded sup-semilattice. Observe that $\mathrm{AuxBottom}(L)$ is auxiliary.

The following propositions are true:

(4) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $\mathrm{AuxBottom}(L) \subseteq A_1$.

(5) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds $\perp_{\langle \mathrm{Aux}(L), \subseteq\rangle} = \mathrm{AuxBottom}(L)$.

Let L be a lower-bounded sup-semilattice. Observe that $\langle \mathrm{Aux}(L), \subseteq\rangle$ is lower-bounded.

One can prove the following propositions:

(6) Let L be a lower-bounded sup-semilattice and let a, b be auxiliary binary relations on L. Then $a \cap b$ is an auxiliary binary relation on L.

(7) Let L be a lower-bounded sup-semilattice and let X be a non empty subset of $\langle \mathrm{Aux}(L), \subseteq\rangle$. Then $\bigcap X$ is an auxiliary binary relation on L.

Let L be a lower-bounded sup-semilattice. One can check that $\langle \mathrm{Aux}(L), \subseteq\rangle$ has g.l.b.'s.

Let L be a lower-bounded sup-semilattice. Observe that $\langle \mathrm{Aux}(L), \subseteq\rangle$ is complete.

Let L be a non empty relation structure, let x be an element of L, and let A_1 be a binary relation on L. The functor $(A_1)-\mathrm{below}(x)$ yielding a subset of L is defined as follows:

(Def.11) $(A_1)-\mathrm{below}(x) = \{y : y$ ranges over elements of L, $\langle y, x\rangle \in A_1\}$.

The functor $(A_1)-\mathrm{above}(x)$ yielding a subset of L is defined as follows:

(Def.12) $(A_1)-\mathrm{above}(x) = \{y : y$ ranges over elements of L, $\langle x, y\rangle \in A_1\}$.

We now state the proposition

(8) Let L be a lower-bounded sup-semilattice, and let x be an element of L, and let A_1 be an auxiliary(i) binary relation on L. Then $(A_1)-\mathrm{below}(x) \subseteq {\downarrow}x$.

Let L be a lower-bounded sup-semilattice, let x be an element of L, and let A_1 be an auxiliary(ii) auxiliary(iii) auxiliary(iv) binary relation on L. Observe that (A_1) −below(x) is directed lower and non empty.

Let L be a lower-bounded sup-semilattice and let A_1 be an auxiliary(ii) auxiliary(iii) auxiliary(iv) binary relation on L. The functor (A_1) −below yielding a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ is defined by:

(Def.13) For every element x of L holds (A_1) −below$(x) = (A_1)$ −below(x).

The following three propositions are true:

(9) Let L be a non empty relation structure, and let A_1 be a binary relation on L, and let a be a set, and let y be an element of L. Then $a \in (A_1)$ −below(y) if and only if $\langle a, y \rangle \in A_1$.

(10) Let L be a sup-semilattice, and let A_1 be a binary relation on L, and let y be an element of L. Then $a \in (A_1)$ −above(y) if and only if $\langle y, a \rangle \in A_1$.

(11) Let L be a lower-bounded sup-semilattice, and let A_1 be an auxiliary binary relation on L, and let x be an element of L. If $A_1 =$ the internal relation of L, then (A_1) −below$(x) = \downarrow x$.

Let L be a non empty poset. The functor MonSet(L) yields a strict relation structure and is defined by the conditions (Def.14).

(Def.14) (i) $a \in$ the carrier of MonSet(L) iff there exists a map s from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ such that $a = s$ and s is monotone and for every element x of L holds $s(x) \subseteq \downarrow x$, and

(ii) for all sets c, d holds $\langle c, d \rangle \in$ the internal relation of MonSet(L) iff there exist maps f, g from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ such that $c = f$ and $d = g$ and $c \in$ the carrier of MonSet(L) and $d \in$ the carrier of MonSet(L) and $f \leq g$.

Next we state two propositions:

(12) Let L be a lower-bounded sup-semilattice. Then MonSet(L) is a full relation substructure of $(\langle \mathrm{Ids}(L), \subseteq \rangle)^{\text{the carrier of } L}$.

(13) Let L be a lower-bounded sup-semilattice, and let A_1 be an auxiliary binary relation on L, and let x, y be elements of L. If $x \leq y$, then (A_1) −below$(x) \subseteq (A_1)$ −below(y).

Let L be a lower-bounded sup-semilattice and let A_1 be an auxiliary binary relation on L. One can verify that (A_1) −below is monotone.

The following proposition is true

(14) Let L be a lower-bounded sup-semilattice and let A_1 be an auxiliary binary relation on L. Then (A_1) −below $\in$ the carrier of MonSet(L).

Let L be a lower-bounded sup-semilattice. One can check that MonSet(L) is non empty.

We now state several propositions:

(15) For every lower-bounded sup-semilattice L holds IdsMap$(L) \in$ the carrier of MonSet(L).

(16) For every lower-bounded sup-semilattice L and for every auxiliary binary relation A_1 on L holds (A_1) −below $\leq$ IdsMap(L).

(17) For every lower-bounded non empty poset L and for every ideal I of L holds $\perp_L \in I$.

(18) For every upper-bounded non empty poset L and for every filter F of L holds $\top_L \in F$.

(19) For every lower-bounded non empty poset L holds $\downarrow(\perp_L) = \{\perp_L\}$.

(20) For every upper-bounded non empty poset L holds $\uparrow(\top_L) = \{\top_L\}$.

In the sequel L is a lower-bounded sup-semilattice, A_1 is an auxiliary binary relation on L, and x is an element of L.

The following propositions are true:

(21) (The carrier of $L) \longmapsto \{\perp_L\}$ is a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$.

(22) (The carrier of $L) \longmapsto \{\perp_L\} \in$ the carrier of MonSet(L).

(23) $\langle$(the carrier of $L) \longmapsto \{\perp_L\}$, (A_1) −below $\rangle \in$ the internal relation of MonSet(L).

Let us consider L. One can verify that MonSet(L) is upper-bounded.

Let us consider L. The functor Rel2Map(L) yielding a map from $\langle \mathrm{Aux}(L), \subseteq \rangle$ into MonSet(L) is defined as follows:

(Def.15) For every A_1 holds $(\mathrm{Rel2Map}(L))(A_1) = (A_1)-\mathrm{below}$.

One can prove the following two propositions:

(24) For all auxiliary binary relations R_1, R_2 on L such that $R_1 \subseteq R_2$ holds $(R_1)-\mathrm{below} \leq (R_2)-\mathrm{below}$.

(25) For all auxiliary binary relations R_1, R_2 on L such that $R_1 \subseteq R_2$ holds $(R_1)-\mathrm{below}(x) \subseteq (R_2)-\mathrm{below}(x)$.

Let us consider L. One can verify that $\mathrm{Rel2Map}(L)$ is monotone.

Let us consider L. The functor $\mathrm{Map2Rel}(L)$ yields a map from $\mathrm{MonSet}(L)$ into $\langle \mathrm{Aux}(L), \subseteq \rangle$ and is defined by the condition (Def.16).

(Def.16) Let s be a set. Suppose $s \in$ the carrier of $\mathrm{MonSet}(L)$. Then there exists an auxiliary binary relation A_1 on L such that

(i) $A_1 = (\mathrm{Map2Rel}(L))(s)$, and

(ii) for all sets x, y holds $\langle x, y \rangle \in A_1$ iff there exist elements x', y' of L and there exists a map s' from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ such that $x' = x$ and $y' = y$ and $s' = s$ and $x' \in s'(y')$.

Let us consider L. Note that $\mathrm{Map2Rel}(L)$ is monotone.

Next we state two propositions:

(26) $\mathrm{Map2Rel}(L) \cdot \mathrm{Rel2Map}(L) = \mathrm{id}_{\mathrm{dom}\,\mathrm{Rel2Map}(L)}$.

(27) $\mathrm{Rel2Map}(L) \cdot \mathrm{Map2Rel}(L) = \mathrm{id}_{(\text{the carrier of } \mathrm{MonSet}(L))}$.

Let us consider L. Observe that $\mathrm{Rel2Map}(L)$ is one-to-one.

We now state three propositions:

(28) $(\mathrm{Rel2Map}(L))^{-1} = \mathrm{Map2Rel}(L)$.

(29) $\mathrm{Rel2Map}(L)$ is isomorphic.

(30) For every complete lattice L and for every element x of L holds $\bigcap\{I : I$ ranges over ideals of L, $x \leq \sup I\} = \downarrow x$.

The scheme *LambdaC'* deals with a non empty relation structure $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

There exists a function f such that $\mathrm{dom}\, f =$ the carrier of $\mathcal{A}$ and for every element x of $\mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ and if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$

for all values of the parameters.

Let L be a semilattice and let I be an ideal of L. The functor $\mathrm{DownMap}(I)$ yielding a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$ is defined as follows:

(Def.17) For every element x of L holds if $x \leq \sup I$, then $(\mathrm{DownMap}(I))(x) = \downarrow x \cap I$ and if $x \not\leq \sup I$, then $(\mathrm{DownMap}(I))(x) = \downarrow x$.

The following two propositions are true:

(31) For every semilattice L and for every ideal I of L holds $\mathrm{DownMap}(I) \in$ the carrier of $\mathrm{MonSet}(L)$.

(32) Let L be an antisymmetric reflexive relation structure with g.l.b.'s, and let x be an element of L, and let D be a non empty lower subset of L. Then $\{x\} \sqcap D = \downarrow x \cap D$.

2. Approximating Relations

Let L be a non empty relation structure and let A_1 be a binary relation on L. We say that A_1 is approximating if and only if:

(Def.18) For every element x of L holds $x = \sup((A_1)-\mathrm{below}(x))$.

Let L be a non empty poset and let m_1 be a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$. We say that m_1 is approximating if and only if:

(Def.19) For every element x of L there exists a subset i_1 of L such that $i_1 = m_1(x)$ and $x = \sup i_1$.

One can prove the following propositions:

(33) For every lower-bounded meet-continuous semilattice L and for every ideal I of L holds $\mathrm{DownMap}(I)$ is approximating.

(34) Every lower-bounded continuous lattice is meet-continuous.

Let us observe that every lower-bounded lattice which is continuous is also meet-continuous.

One can prove the following proposition

(35) For every lower-bounded continuous lattice L and for every ideal I of L holds $\mathrm{DownMap}(I)$ is approximating.

Let L be a non empty reflexive antisymmetric relation structure. One can verify that (L) −waybelow is auxiliary(i).

Let L be a non empty reflexive transitive relation structure. Observe that (L) −waybelow is auxiliary(ii).

Let L be a poset with l.u.b.'s. Observe that (L) −waybelow is auxiliary(iii).

Let L be an inf-complete non empty poset. One can check that (L) −waybelow is auxiliary(iii).

Let L be a lower-bounded antisymmetric reflexive non empty relation structure. Note that (L) −waybelow is auxiliary(iv).

One can prove the following two propositions:

(36) For every complete lattice L and for every element x of L holds $((L)\,\text{−waybelow})\,\text{−below}(x) = {\downarrow}x$.

(37) For every lattice L holds $\mathrm{IntRel}(L)$ is approximating.

Let L be a lower-bounded continuous lattice. Observe that (L) −waybelow is approximating.

Let L be a complete lattice. One can verify that there exists an auxiliary binary relation on L which is approximating.

Let L be a complete lattice. The functor $\mathrm{App}(L)$ is defined as follows:

(Def.20) $a \in \mathrm{App}(L)$ iff a is an approximating auxiliary binary relation on L.

Let L be a complete lattice. Note that $\mathrm{App}(L)$ is non empty.

Next we state three propositions:

(38) Let L be a complete lattice and let m_1 be a map from L into $\langle \mathrm{Ids}(L), \subseteq \rangle$. Suppose m_1 is approximating and $m_1 \in$ the carrier of $\mathrm{MonSet}(L)$. Then there exists an approximating auxiliary binary relation A_1 on L such that $A_1 = (\mathrm{Map2Rel}(L))(m_1)$.

(39) For every complete lattice L and for every element x of L holds $\bigcap\{(\mathrm{DownMap}(I))(x) : I$ ranges over ideals of $L\} = {\downarrow}x$.

(40) Let L be a lower-bounded meet-continuous lattice and let x be an element of L. Then $\bigcap\{(A_1)\,\text{−below}(x) : A_1$ ranges over auxiliary binary relations on L, $A_1 \in \mathrm{App}(L)\} = {\downarrow}x$.

In the sequel L will denote a complete lattice.

One can prove the following propositions:

(41) L is continuous if and only if for every approximating auxiliary binary relation R on L holds (L) −waybelow $\subseteq R$ and (L) −waybelow is approximating.

(42) L is continuous if and only if the following conditions are satisfied:
 (i) L is meet-continuous, and
 (ii) there exists an approximating auxiliary binary relation R on L such that for every approximating auxiliary binary relation R' on L holds $R \subseteq R'$.

Let L be a non empty relation structure and let A_1 be a binary relation on L. We say that A_1 satisfies SI if and only if:

(Def.21) For all elements x, z of L such that $\langle x, z \rangle \in A_1$ and $x \neq z$ there exists an element y of L such that $\langle x, y \rangle \in A_1$ and $\langle y, z \rangle \in A_1$ and $x \neq y$.

We introduce A_1 is filtered-infs-preserving as a synonym of A_1 satisfies SI.

Let L be a non empty relation structure and let A_1 be a binary relation on L. We say that A_1 satisfies INT if and only if:

(Def.22) For all elements x, z of L such that $\langle x, z\rangle \in A_1$ there exists an element y of L such that $\langle x, y\rangle \in A_1$ and $\langle y, z\rangle \in A_1$.

We introduce A_1 is directed-sups-inheriting as a synonym of A_1 satisfies INT.

The following propositions are true:

(43) Let L be a non empty relation structure, and let A_1 be a binary relation on L, and let x, z be elements of L. If $\langle x, z\rangle \in A_1$ and $x = z$, then there exists an element y of L such that $\langle x, y\rangle \in A_1$ and $\langle y, z\rangle \in A_1$.

(44) Let L be a non empty relation structure and let A_1 be a binary relation on L. If A_1 satisfies SI , then A_1 satisfies INT .

Let L be a non empty relation structure. Note that every binary relation on L which satisfies SI satisfies also INT.

In the sequel A_1 will denote an auxiliary binary relation on L and x, y, z will denote elements of L.

One can prove the following four propositions:

(45) Let A_1 be an approximating auxiliary binary relation on L. If $x \not\leq y$, then there exists an element u of L such that $\langle u, x\rangle \in A_1$ and $u \not\leq y$.

(46) Let R be an approximating auxiliary binary relation on L. If $\langle x, z\rangle \in R$ and $x \neq z$, then there exists y such that $x \leq y$ and $\langle y, z\rangle \in R$ and $x \neq y$.

(47) Let R be an approximating auxiliary binary relation on L. Suppose $x \ll z$ and $x \neq z$. Then there exists an element y of L such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ and $x \neq y$.

(48) For every lower-bounded continuous lattice L holds (L) −waybelow satisfies SI .

Let L be a lower-bounded continuous lattice. Note that (L) −waybelow satisfies SI.

We now state two propositions:

(49) Let L be a lower-bounded continuous lattice and let x, y be elements of L. If $x \ll y$, then there exists an element x' of L such that $x \ll x'$ and $x' \ll y$.

(50) Let L be a lower-bounded continuous lattice and let x, y be elements of L. Then $x \ll y$ if and only if for every non empty directed subset D of L such that $y \leq \sup D$ there exists an element d of L such that $d \in D$ and $x \ll d$.

3. Exercises

Let L be a relation structure, let X be a subset of L, and let R be a binary relation on the carrier of L. We say that X is directed wrt R if and only if:

(Def.23) For all elements x, y of L such that $x \in X$ and $y \in X$ there exists an element z of L such that $z \in X$ and $\langle x, z\rangle \in R$ and $\langle y, z\rangle \in R$.

One can prove the following proposition

(51) Let L be a relation structure and let X be a subset of L. Suppose X is directed wrt the internal relation of L. Then X is directed.

Let L be a relation structure, let X be a set, let x be an element of L, and let R be a binary relation on the carrier of L. We say that x is maximal wrt X, R if and only if:

(Def.24) $x \in X$ and it is not true that there exists an element y of L such that $y \in X$ and $y \neq x$ and $\langle x, y\rangle \in R$.

Let L be a relation structure, let X be a set, and let x be an element of L. We say that x is maximal in X if and only if:

(Def.25) x is maximal wrt X, the internal relation of L.

The following proposition is true

(52) Let L be a relation structure, and let X be a set, and let x be an element of L. Then x is maximal in X if and only if the following conditions are satisfied:

(i) $x \in X$, and

(ii) it is not true that there exists an element y of L such that $y \in X$ and $x < y$.

Let L be a relation structure, let X be a set, let x be an element of L, and let R be a binary relation on the carrier of L. We say that x is minimal wrt X, R if and only if:

(Def.26) $x \in X$ and it is not true that there exists an element y of L such that $y \in X$ and $y \neq x$ and $\langle y, x \rangle \in R$.

Let L be a relation structure, let X be a set, and let x be an element of L. We say that x is minimal in X if and only if:

(Def.27) x is minimal wrt X, the internal relation of L.

The following propositions are true:

(53) Let L be a relation structure, and let X be a set, and let x be an element of L. Then x is minimal in X if and only if the following conditions are satisfied:

(i) $x \in X$, and

(ii) it is not true that there exists an element y of L such that $y \in X$ and $x > y$.

(54) If A_1 satisfies SI , then for every x such that there exists y such that y is maximal wrt (A_1) −below(x), A_1 holds $\langle x, x \rangle \in A_1$.

(55) If for every x such that there exists y such that y is maximal wrt (A_1) −below(x), A_1 holds $\langle x, x \rangle \in A_1$, then A_1 satisfies SI .

(56) If A_1 satisfies INT , then for every x holds (A_1) −below(x) is directed wrt A_1.

(57) If for every x holds (A_1) −below(x) is directed wrt A_1, then A_1 satisfies INT .

(58) Let R be an approximating auxiliary binary relation on L. If R satisfies INT , then R satisfies SI .

Let us consider L. Observe that every approximating auxiliary binary relation on L which satisfies INT satisfies also SI.

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The Equational Characterization of Continuous Lattices [1]

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Summary. The class of continuous lattices can be characterized by infinitary equations. Therefore, it is closed under the formation of subalgebras and homomorphic images. Following the terminology of [18] we introduce a continuous lattice subframe to be a sublattice closed under the formation of arbitrary infs and directed sups. This notion corresponds with a subalgebra of a continuous lattice in [15].

The class of completely distributive lattices is also introduced in the paper. Such lattices are complete and satisfy the most restrictive type of general distributivity law. Obviously each completely distributive lattice is a Heyting algebra. It was hard to find the best Mizar implementation of the complete distributivity equational condition (denoted by CD in [15]). The powerful and well developed Many Sorted Theory gives the most convenient way of this formalization. A set double indexed by K, introduced in the paper, correspond with a family $\{x_{j,k} : j \in J, k \in K(j)\}$. It is defined to be a suitable many sorted function. Two special functors: Sups and Infs as counterparts of Sup and Inf respectively, introduced in [35], are also defined. Originally the equation in Definition 2.4 of [15, p. 58] looks as follows:

$$\bigwedge_{j\in J}\bigvee_{k\in K(j)}x_{j,k} = \bigvee_{f\in M}\bigwedge_{j\in J}x_{j,f(j)},$$

where M is the set of functions defined on J with values $f(j) \in K(j)$. The Mizar implementation can be found below.

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The articles [30], [33], [34], [13], [14], [31], [9], [10], [4], [2], [27], [32], [3], [6], [26], [12], [8], [29], [22], [23], [28], [20], [24], [25], [19], [21], [16], [1], [17], [35], [11], [5], and [7] provide the notation and terminology for this paper.

1. The Continuity of Lattices

In this paper x, y will be arbitrary, X will be a set, and L will be an up-complete semilattice.

One can prove the following propositions:

(1) L is continuous if and only if for every element x of L holds $\downarrow x$ is an ideal of L and $x \leq \sup \downarrow x$ and for every ideal I of L such that $x \leq \sup I$ holds $\downarrow x \subseteq I$.

(2) L is continuous if and only if for every element x of L there exists an ideal I of L such that $x \leq \sup I$ and for every ideal J of L such that $x \leq \sup J$ holds $I \subseteq J$.

(3) For every continuous lower-bounded sup-semilattice L holds SupMap(L) has a lower adjoint.

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(4) For every up-complete lower-bounded lattice L such that $\mathrm{SupMap}(L)$ is upper adjoint holds L is continuous.

(5) For every complete semilattice L such that $\mathrm{SupMap}(L)$ is infs-preserving and sups-preserving holds $\mathrm{SupMap}(L)$ has a lower adjoint.

Let J, D be sets and let K be a many sorted set indexed by J. A set of elements of D double indexed by K is a many sorted function from K into $J \longmapsto D$.

Let J be a set, let K be a many sorted set indexed by J, and let S be a 1-sorted structure. A set of elements of S double indexed by K is a set of elements of the carrier of S double indexed by K.

One can prove the following proposition

(6) Let J, D be sets, and let K be a many sorted set indexed by J, and let F be a set of elements of D double indexed by K, and let j be arbitrary. If $j \in J$, then $F(j)$ is a function from $K(j)$ into D.

Let J, D be non empty sets, let K be a many sorted set indexed by J, let F be a set of elements of D double indexed by K, and let j be an element of J. Then $F(j)$ is a function from $K(j)$ into D.

Let J, D be non empty sets, let K be a non-empty many sorted set indexed by J, let F be a set of elements of D double indexed by K, and let j be an element of J. Observe that $\mathrm{rng}\, F(j)$ is non empty.

Let J be a set, let D be a non empty set, and let K be a non-empty many sorted set indexed by J. Note that every set of elements of D double indexed by K is non-empty.

Next we state four propositions:

(7) For every function yielding function F and for arbitrary f such that $f \in \mathrm{dom}\,\mathrm{Frege}(F)$ holds f is a function.

(8) For every function yielding function F and for every function f such that $f \in \mathrm{dom}\,\mathrm{Frege}(F)$ holds $\mathrm{dom}\, f = \mathrm{dom}\, F$ and $\mathrm{dom}\, F = \mathrm{dom}(\mathrm{Frege}(F))(f)$.

(9) Let F be a function yielding function and let f be a function. Suppose $f \in \mathrm{dom}\,\mathrm{Frege}(F)$. Let i be arbitrary. If $i \in \mathrm{dom}\, F$, then $f(i) \in \mathrm{dom}\, F(i)$ and $(\mathrm{Frege}(F))(f)(i) = F(i)(f(i))$ and $F(i)(f(i)) \in \mathrm{rng}(\mathrm{Frege}(F))(f)$.

(10) Let J, D be sets, and let K be a many sorted set indexed by J, and let F be a set of elements of D double indexed by K, and let f be a function. If $f \in \mathrm{dom}\,\mathrm{Frege}(F)$, then $(\mathrm{Frege}(F))(f)$ is a function from J into D.

Let J be a set, let S be a non empty 1-sorted structure, and let F be a function from J into the carrier of S. Then $\mathrm{rng}\, F$ is a subset of S.

Let f be a non-empty function. Note that $\mathrm{dom}_\kappa f(\kappa)$ is non-empty.

Let J, D be sets, let K be a many sorted set indexed by J, and let F be a set of elements of D double indexed by K. Then $\mathrm{Frege}(F)$ is a set of elements of D double indexed by $\prod(\mathrm{dom}_\kappa F(\kappa)) \longmapsto J$.

Let J, D be non empty sets, let K be a non-empty many sorted set indexed by J, let F be a set of elements of D double indexed by K, let G be a set of elements of D double indexed by $\prod(\mathrm{dom}_\kappa F(\kappa)) \longmapsto J$, and let f be an element of $\prod(\mathrm{dom}_\kappa F(\kappa))$. Then $G(f)$ is a function from J into D.

Let L be a non empty relation structure and let F be a function yielding function. The functor $\underline{\bigsqcup}_L F$ yielding a function from $\mathrm{dom}\, F$ into the carrier of L is defined by:

(Def.1) For every x such that $x \in \mathrm{dom}\, F$ holds $(\underline{\bigsqcup}_L F)(x) = \bigsqcup_L F(x)$.

The functor $\overline{\bigsqcap}_L F$ yielding a function from $\mathrm{dom}\, F$ into the carrier of L is defined as follows:

(Def.2) For every x such that $x \in \mathrm{dom}\, F$ holds $(\overline{\bigsqcap}_L F)(x) = \bigsqcap_L F(x)$.

Let J be a set, let K be a many sorted set indexed by J, let L be a non empty relation structure, and let F be a set of elements of L double indexed by K. We introduce $\mathrm{Sups}(F)$ as a synonym of $\underline{\bigsqcup}_L F$. We introduce $\mathrm{Infs}(F)$ as a synonym of $\overline{\bigsqcap}_L F$.

Let I, J be sets, let L be a non empty relation structure, and let F be a set of elements of L double indexed by $I \longmapsto J$. We introduce $\mathrm{Sups}(F)$ as a synonym of $\underline{\bigsqcup}_L F$. We introduce $\mathrm{Infs}(F)$ as a synonym

of $\overline{\bigsqcap}_L F$.

The following propositions are true:

(11) Let L be a non empty relation structure and let F, G be function yielding functions. If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $\bigsqcup_L F(x) = \bigsqcup_L G(x)$, then $\underline{\bigsqcup}_L F = \underline{\bigsqcup}_L G$.

(12) Let L be a non empty relation structure and let F, G be function yielding functions. If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $\bigsqcap_L F(x) = \bigsqcap_L G(x)$, then $\overline{\bigsqcap}_L F = \overline{\bigsqcap}_L G$.

(13) Let L be a non empty relation structure and let F be a function yielding function. Then

(i) $y \in \operatorname{rng} \underline{\bigsqcup}_L F$ iff there exists x such that $x \in \operatorname{dom} F$ and $y = \bigsqcup_L F(x)$, and

(ii) $y \in \operatorname{rng} \overline{\bigsqcap}_L F$ iff there exists x such that $x \in \operatorname{dom} F$ and $y = \bigsqcap_L F(x)$.

(14) Let L be a non empty relation structure, and let J be a non empty set, and let K be a many sorted set indexed by J, and let F be a set of elements of L double indexed by K. Then

(i) $x \in \operatorname{rng} \operatorname{Sups}(F)$ iff there exists an element j of J such that $x = \operatorname{Sup}(F(j))$, and

(ii) $x \in \operatorname{rng} \operatorname{Infs}(F)$ iff there exists an element j of J such that $x = \operatorname{Inf}(F(j))$.

Let J be a non empty set, let K be a many sorted set indexed by J, let L be a non empty relation structure, and let F be a set of elements of L double indexed by K. One can check that $\operatorname{rng} \operatorname{Sups}(F)$ is non empty and $\operatorname{rng} \operatorname{Infs}(F)$ is non empty.

For simplicity we follow a convention: L will be a complete lattice, a, b, c will be elements of L, J will be a non empty set, and K will be a non-empty many sorted set indexed by J.

We now state four propositions:

(15) Let F be a function yielding function. If for every function f such that $f \in \operatorname{dom} \operatorname{Frege}(F)$ holds $\bigsqcap_L (\operatorname{Frege}(F))(f) \leq a$, then $\operatorname{Sup}(\overline{\bigsqcap}_L \operatorname{Frege}(F)) \leq a$.

(16) For every set F of elements of L double indexed by K holds $\operatorname{Inf}(\operatorname{Sups}(F)) \geq \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(F)))$.

(17) If L is continuous and for every c such that $c \ll a$ holds $c \leq b$, then $a \leq b$.

(18) L is continuous if and only if for all J, K and for every set F of elements of L double indexed by K such that for every element j of J holds $\operatorname{rng} F(j)$ is directed holds $\operatorname{Inf}(\operatorname{Sups}(F)) = \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(F)))$.

Let J, K, D be non empty sets and let F be a function from $[\!: J, K :\!]$ into D. Then $\operatorname{curry} F$ is a set of elements of D double indexed by $J \longmapsto K$.

We follow the rules: J, K, D denote non empty sets, j denotes an element of J, and k denotes an element of K.

The following four propositions are true:

(19) For every function F from $[\!: J, K :\!]$ into D holds $\operatorname{dom} \operatorname{curry} F = J$ and $\operatorname{dom}(\operatorname{curry} F)(j) = K$ and $F(\langle j, k \rangle) = (\operatorname{curry} F)(j)(k)$.

(20) L is continuous if and only if for all non empty sets J, K and for every function F from $[\!: J, K :\!]$ into the carrier of L such that for every element j of J holds $\operatorname{rng}(\operatorname{curry} F)(j)$ is directed holds $\operatorname{Inf}(\operatorname{Sups}(\operatorname{curry} F)) = \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(\operatorname{curry} F)))$.

(21) Let F be a function from $[\!: J, K :\!]$ into the carrier of L and let X be a subset of L. Suppose $X = \{a : a$ ranges over elements of L, $\bigvee_{f : \text{non-emptymany sorted set indexed by } J} f \in (\operatorname{Fin} K)^J \wedge \bigvee_{G : \text{set of elements of } L \text{ double indexed by } f} \bigwedge_{j,x} x \in f(j) \Rightarrow G(j)(x) = F(\langle j, x \rangle) \wedge a = \operatorname{Inf}(\operatorname{Sups}(G))\}$. Then $\operatorname{Inf}(\operatorname{Sups}(\operatorname{curry} F)) \geq \sup X$.

(22) L is continuous if and only if for all J, K and for every function F from $[\!: J, K :\!]$ into the carrier of L and for every subset X of L such that $X = \{a : a$ ranges over elements of L, $\bigvee_{f : \text{non-emptymany sorted set indexed by } J} f \in (\operatorname{Fin} K)^J \wedge \bigvee_{G : \text{set of elements of } L \text{ double indexed by } f} \bigwedge_{j,x} x \in f(j) \Rightarrow G(j)(x) = F(\langle j, x \rangle) \wedge a = \operatorname{Inf}(\operatorname{Sups}(G))\}$ holds $\operatorname{Inf}(\operatorname{Sups}(\operatorname{curry} F)) = \sup X$.

2. Completely-Distributive Lattices

Let L be a non empty relation structure. We say that L is completely-distributive if and only if the conditions (Def.3) are satisfied.

(Def.3) (i) L is complete, and

(ii) for every non empty set J and for every non-empty many sorted set K indexed by J and for every set F of elements of L double indexed by K holds $\mathrm{Inf}(\mathrm{Sups}(F)) = \mathrm{Sup}(\mathrm{Infs}(\mathrm{Frege}(F)))$.

In the sequel J denotes a non empty set and K denotes a non-empty many sorted set indexed by J.

Let us mention that every non empty poset which is trivial is also completely-distributive.

Let us observe that there exists a lattice which is completely-distributive.

Next we state the proposition

(23) Every completely-distributive lattice is continuous.

Let us note that every lattice which is completely-distributive is also complete and continuous.

The following propositions are true:

(24) Let L be a non empty antisymmetric transitive relation structure with g.l.b.'s, and let x be an element of L, and let X, Y be subsets of L. Suppose $\sup X$ exists in L and $\sup Y$ exists in L and $Y = \{x \sqcap y : y$ ranges over elements of L, $y \in X\}$. Then $x \sqcap \sup X \geq \sup Y$.

(25) Let L be a completely-distributive lattice, and let X be a subset of L, and let x be an element of L. Then $x \sqcap \sup X = \bigsqcup_L \{x \sqcap y : y$ ranges over elements of L, $y \in X\}$.

Let us mention that every lattice which is completely-distributive is also Heyting.

3. SubFrames of Continuous Lattices

Let L be a non empty relation structure. A continuous subframe of L is an infs-inheriting directed-sups-inheriting non empty full relation substructure of L.

We now state three propositions:

(26) Let F be a set of elements of L double indexed by K. If for every element j of J holds $\mathrm{rng}\, F(j)$ is directed, then $\mathrm{rng}\, \mathrm{Infs}(\mathrm{Frege}(F))$ is directed.

(27) If L is continuous, then every continuous subframe of L is a continuous lattice.

(28) For every non empty poset S such that there exists map from L into S which is infs-preserving and onto holds S is a complete lattice.

Let J be a set and let y be arbitrary. We introduce $J \Longrightarrow y$ as a synonym of $J \longmapsto y$.

Let J be a set and let y be arbitrary. Then $J \longmapsto y$ is a many sorted set indexed by J. We introduce $J \Longrightarrow y$ as a synonym of $J \longmapsto y$.

Let A, B, J be sets and let f be a function from A into B. Then $J \Longrightarrow f$ is a many sorted function from $J \longmapsto A$ into $J \longmapsto B$.

We now state four propositions:

(29) Let A, B be sets, and let f be a function from A into B, and let g be a function from B into A. If $g \cdot f = \mathrm{id}_A$, then $(J \Longrightarrow g) \circ (J \Longrightarrow f) = \mathrm{id}_{(J \longmapsto A)}$.

(30) Let J, A be non empty sets, and let B be a set, and let K be a many sorted set indexed by J, and let F be a set of elements of A double indexed by K, and let f be a function from A into B. Then $(J \Longrightarrow f) \circ F$ is a set of elements of B double indexed by K.

(31) Let J, A, B be non empty sets, and let K be a many sorted set indexed by J, and let F be a set of elements of A double indexed by K, and let f be a function from A into B. Then $\mathrm{dom}_\kappa (J \Longrightarrow f) \circ F(\kappa) = \mathrm{dom}_\kappa F(\kappa)$.

(32) Suppose L is continuous. Let S be a non empty poset. Suppose there exists map from L into S which is infs-preserving directed-sups-preserving and onto. Then S is a continuous lattice.

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Scott Topology [1]

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Summary. In the article we continue the formalization in Mizar of [16]. We work with structures of the form

$$L = \langle C, \leq, \tau \rangle,$$

where C is the carrier of the structure, $\leq$ - an ordering relation on C and τ a family of subsets of C.

When $\langle C, \leq \rangle$ is a complete lattice we say that L is Scott if τ is the Scott topology of $\langle C, \leq \rangle$.

We define the Scott convergence (lim inf convergence). Following [16] we prove that in the case of a continuous lattice $\langle C, \leq \rangle$ the Scott convergence is topological, i.e. enjoys the properties: CONSTANT, SUBNET, DIVERGENCE, ITERATED LIMITS. We formalize the theorem that if the Scott convergence has the ITERATED LIMITS property, the $\langle C, \leq \rangle$ is continuous.

The facts proved in the article cover pages 98-105 of the book.

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1. Preliminaries

The scheme *Irrel* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(u) : u$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[u]\} = \{\mathcal{F}(i,v) : i$ ranges over elements of $\mathcal{B}$, v ranges over elements of $\mathcal{A}$, $\mathcal{P}[v]\}$

provided the following condition is satisfied:

- For every element i of $\mathcal{B}$ and for every element u of $\mathcal{A}$ holds $\mathcal{F}(u) = \mathcal{F}(i,u)$.

One can prove the following propositions:

(1) Let L be a complete non empty lattice and let X, Y be subsets of the carrier of L. If Y is coarser than X, then $\bigsqcap_L X \leq \bigsqcap_L Y$.

(2) Let L be a complete non empty lattice and let X, Y be subsets of the carrier of L. If X is finer than Y, then $\bigsqcup_L X \leq \bigsqcup_L Y$.

(3) Let T be a relation structure, and let A be an upper subset of T, and let B be a directed subset of T. Then $A \cap B$ is directed.

Let T be a reflexive non empty relation structure. One can check that there exists a subset of T which is non empty directed and finite.

Next we state the proposition

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(4) For every non empty poset T with l.u.b.'s and for every non empty directed finite subset D of T holds $\sup D \in D$.

Let us observe that there exists a relation structure which is trivial reflexive transitive non empty antisymmetric finite and strict and has l.u.b.'s.

One can verify that there exists a 1-sorted structure which is finite non empty and strict.

Let T be a finite 1-sorted structure. One can verify that every subset of T is finite.

Let R be a relation structure. One can check that $\emptyset_R$ is lower and upper.

Let R be a trivial non empty relation structure. Observe that every subset of R is upper.

Next we state two propositions:

(5) Let T be a non empty relation structure, and let x be an element of T, and let A be an upper subset of T. If $x \notin A$, then A misses $\downarrow x$.

(6) Let T be a non empty relation structure, and let x be an element of T, and let A be a lower subset of T. If $x \in A$, then $\downarrow x \subseteq A$.

2. Scott Topology

Let T be a non empty reflexive relation structure and let S be a subset of T. We say that S is inaccessible by directed joins if and only if:

(Def. 1) For every non empty directed subset D of T such that $\sup D \in S$ holds D meets S.

We introduce S is inaccessible as a synonym of S is inaccessible by directed joins. We say that S is closed under directed sups if and only if:

(Def. 2) For every non empty directed subset D of T such that $D \subseteq S$ holds $\sup D \in S$.

We introduce S is directly closed as a synonym of S is closed under directed sups. We say that S has the property (S) if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let D be a non empty directed subset of T. Suppose $\sup D \in S$. Then there exists an element y of T such that $y \in D$ and for every element x of T such that $x \in D$ and $x \geq y$ holds $x \in S$.

Let T be a non empty reflexive relation structure. Observe that $\emptyset_T$ is directly closed and has the property (S).

Let T be a non empty reflexive relation structure. One can verify that there exists a subset of T which is directly closed and has the property (S).

Let T be a non empty reflexive relation structure and let S be a subset of T with the property (S). Observe that $-S$ is directly closed.

Let T be a reflexive non empty TopRelStr. We say that T is Scott if and only if:

(Def. 4) For every subset S of T holds S is open iff S is inaccessible and upper.

Let T be a reflexive transitive antisymmetric non empty finite relation structure with l.u.b.'s. Observe that every subset of T is inaccessible.

Let T be a reflexive transitive antisymmetric non empty finite TopRelStr with l.u.b.'s. Let us observe that T is Scott if and only if:

(Def. 5) For every subset S of T holds S is open iff S is upper.

Let us note that there exists a non empty strict TopLattice which is trivial complete and Scott.

Let T be a non empty reflexive relation structure. Note that Ω_T is directly closed and inaccessible.

Let T be a non empty reflexive relation structure. Note that there exists a subset of T which is directly closed lower inaccessible and upper.

Let T be a complete non empty TopLattice and let S be an inaccessible subset of T. Observe that $-S$ is directly closed.

Let T be a non empty reflexive relation structure and let S be a directly closed subset of T. Note that $-S$ is inaccessible.

Next we state several propositions:

(7) Let T be a complete Scott non empty TopLattice and let S be a subset of T. Then S is closed if and only if S is directly closed and lower.

(8) For every complete non empty TopLattice T and for every element x of T holds $\downarrow x$ is directly closed.

(9) For every complete Scott non empty TopLattice T and for every element x of T holds $\overline{\{x\}} = \downarrow x$.

(10) Every complete Scott non empty TopLattice is a T_0-space.

(11) For every complete Scott non empty TopLattice T and for every element x of T holds $\downarrow x$ is closed.

(12) For every complete Scott non empty TopLattice T and for every element x of T holds $-\downarrow x$ is open.

(13) Let T be a complete Scott non empty TopLattice, and let x be an element of T, and let A be an upper subset of T. If $x \notin A$, then $-\downarrow x$ is a neighbourhood of A.

(14) Let T be a complete Scott non empty TopLattice and let S be an upper subset of T. Then there exists a family F of subsets of T such that $S = \bigcap F$ and for every subset X of T such that $X \in F$ holds X is a neighbourhood of S.

(15) Let T be a Scott non empty TopLattice and let S be a subset of T. Then S is open if and only if S is upper and has the property (S).

Let T be a complete non empty TopLattice. Note that every subset of T which is lower has the property (S).

The following proposition is true

(16) Let T be a non empty transitive reflexive TopRelStr. Suppose the topology of $T = \{S : S$ ranges over subsets of T, S has the property (S)$\}$. Then T is topological space-like.

3. Scott Convergence

In the sequel R will denote a non empty relation structure, N will denote a net in R, and i, j will denote elements of the carrier of N.

Let us consider R, N. The functor $\liminf N$ yields an element of R and is defined by:

(Def. 6) $\liminf N = \bigsqcup_R \{\bigsqcap_R \{N(i) : i \geq j\} : j$ ranges over elements of the carrier of $N\}$.

Let R be a reflexive non empty relation structure, let N be a net in R, and let p be an element of the carrier of R. We say that p is S-limit of N if and only if:

(Def. 7) $p \leq \liminf N$.

Let R be a reflexive non empty relation structure. The functor Scott-Convergence(R) yielding a Convergence-Class of R is defined by the condition (Def. 8).

(Def. 8) Let N be a strict net in R. Suppose $N \in \mathrm{NetUniv}(R)$. Let p be an element of the carrier of R. Then $\langle N, p \rangle \in$ Scott-Convergence(R) if and only if p is S-limit of N.

The following propositions are true:

(17) Let R be a non empty complete lattice, and let N be a net in R, and let p, q be elements of the carrier of R. If p is S-limit of N and N is eventually in $\downarrow q$, then $p \leq q$.

(18) Let R be a non empty complete lattice, and let N be a net in R, and let p, q be elements of the carrier of R. If N is eventually in $\uparrow q$, then $\liminf N \geq q$.

Let R be a reflexive non empty relation structure and let N be a non empty net structure over R Let us observe that N is monotone if and only if:

(Def. 9) For all elements i, j of the carrier of N such that $i \leq j$ holds $N(i) \leq N(j)$.

Let R be a non empty relation structure, let S be a non empty set, and let f be a function from S into the carrier of R. The functor NetStr(S, f) yields a strict non empty net structure over R and is defined by the conditions (Def. 10).

(Def. 10) (i) The carrier of $\mathrm{NetStr}(S, f) = S$,

(ii) the mapping of $\mathrm{NetStr}(S, f) = f$, and

(iii) for all elements i, j of $\mathrm{NetStr}(S, f)$ holds $i \leq j$ iff $(\mathrm{NetStr}(S, f))(i) \leq (\mathrm{NetStr}(S, f))(j)$.

We now state two propositions:

(19) Let L be a non empty 1-sorted structure and let N be a non empty net structure over L. Then rng (the mapping of N) $= \{N(i) : i$ ranges over elements of the carrier of $N\}$.

(20) Let R be a non empty relation structure, and let S be a non empty set, and let f be a function from S into the carrier of R. If $\mathrm{rng}\, f$ is directed, then $\mathrm{NetStr}(S, f)$ is directed.

Let R be a non empty relation structure, let S be a non empty set, and let f be a function from S into the carrier of R. Note that $\mathrm{NetStr}(S, f)$ is monotone.

Let R be a transitive non empty relation structure, let S be a non empty set, and let f be a function from S into the carrier of R. Note that $\mathrm{NetStr}(S, f)$ is transitive.

Let R be a reflexive non empty relation structure, let S be a non empty set, and let f be a function from S into the carrier of R. Observe that $\mathrm{NetStr}(S, f)$ is reflexive.

The following proposition is true

(21) Let R be a non empty transitive relation structure, and let S be a non empty set, and let f be a function from S into the carrier of R. If $S \subseteq$ the carrier of R and $\mathrm{NetStr}(S, f)$ is directed, then $\mathrm{NetStr}(S, f) \in \mathrm{NetUniv}(R)$.

Let R be a non empty lattice. Observe that there exists a net in R which is monotone reflexive and strict.

One can prove the following propositions:

(22) For every non empty complete lattice R and for every monotone reflexive net N in R holds $\liminf N = \sup N$.

(23) For every complete non empty lattice R and for every constant net N in R holds $\mathrm{thevalueof}(N) = \liminf N$.

(24) For every complete non empty lattice R and for every constant net N in R holds $\mathrm{thevalueof}(N)$ is S-limit of N.

Let S be a non empty 1-sorted structure and let e be an element of the carrier of S. The functor $\mathrm{NetStr}(e)$ yields a strict net structure over S and is defined by:

(Def. 11) The carrier of $\mathrm{NetStr}(e) = \{e\}$ and the internal relation of $\mathrm{NetStr}(e) = \{\langle e, e\rangle\}$ and the mapping of $\mathrm{NetStr}(e) = \mathrm{id}_{\{e\}}$.

Let S be a non empty 1-sorted structure and let e be an element of the carrier of S. Observe that $\mathrm{NetStr}(e)$ is non empty.

The following two propositions are true:

(25) Let S be a non empty 1-sorted structure, and let e be an element of the carrier of S, and let x be an element of $\mathrm{NetStr}(e)$. Then $x = e$.

(26) Let S be a non empty 1-sorted structure, and let e be an element of the carrier of S, and let x be an element of $\mathrm{NetStr}(e)$. Then $(\mathrm{NetStr}(e))(x) = e$.

Let S be a non empty 1-sorted structure and let e be an element of the carrier of S. Note that $\mathrm{NetStr}(e)$ is reflexive transitive directed and antisymmetric.

Next we state several propositions:

(27) Let S be a non empty 1-sorted structure, and let e be an element of the carrier of S, and let X be a set. Then $\mathrm{NetStr}(e)$ is eventually in X if and only if $e \in X$.

(28) Let S be a reflexive antisymmetric non empty relation structure and let e be an element of the carrier of S. Then $e = \liminf \mathrm{NetStr}(e)$.

(29) For every non empty reflexive relation structure S and for every element e of the carrier of S holds $\mathrm{NetStr}(e) \in \mathrm{NetUniv}(S)$.

(30) Let R be a non empty complete lattice, and let Z be a net in R, and let D be a subset of R. Suppose $D = \{\bigsqcap_R\{Z(k) : k$ ranges over elements of the carrier of Z, $k \geq j\} : j$ ranges over elements of the carrier of $Z\}$. Then D is non empty and directed.

(31) Let L be a non empty complete lattice and let S be a subset of L. Then $S \in$ the topology of ConvergenceSpace(Scott-Convergence(L)) if and only if S is inaccessible and upper.

(32) For every non empty complete Scott TopLattice T holds the topological structure of $T =$ ConvergenceSpace(Scott-Convergence(T)).

(33) Let T be a non empty complete TopLattice. Suppose the topological structure of $T =$ ConvergenceSpace(Scott-Convergence(T)). Let S be a subset of T. Then S is open if and only if S is inaccessible and upper.

(34) For every non empty complete TopLattice T such that the topological structure of $T =$ ConvergenceSpace(Scott-Convergence(T)) holds T is Scott.

Let R be a complete non empty lattice. Note that Scott-Convergence(R) is (CONSTANTS).

Let R be a complete non empty lattice. One can verify that Scott-Convergence(R) is (SUBNETS).

One can prove the following proposition

(35) Let S be a non empty 1-sorted structure, and let N be a net in S, and let X be a set, and let M be a subnet of N. If $M = N^{-1} X$, then for every element i of the carrier of M holds $M(i) \in X$.

Let L be a non empty complete lattice. The functor sigma(L) yields a family of subsets of L and is defined by:

(Def. 12) sigma(L) = the topology of ConvergenceSpace(Scott-Convergence(L)).

Next we state two propositions:

(36) For every continuous complete Scott TopLattice L and for every element x of L holds $\Uparrow x$ is open.

(37) For every non empty complete TopLattice T such that the topology of $T =$ sigma(T) holds T is Scott.

Let R be a continuous non empty complete lattice. One can check that Scott-Convergence(R) is topological.

One can prove the following propositions:

(38) Let T be a continuous non empty complete Scott TopLattice, and let x be an element of the carrier of T, and let N be a net in T. If $N \in$ NetUniv(T), then x is S-limit of N iff $x \in \mathrm{Lim}\, N$.

(39) For every complete non empty poset L such that Scott-Convergence(L) is (ITERATED LIMITS) holds L is continuous.

(40) For every complete Scott non empty TopLattice T holds T is continuous iff Convergence(T) = Scott-Convergence(T).

(41) For every complete Scott non empty TopLattice T and for every upper subset S of T such that S is Open holds S is open.

(42) Let L be a non empty relation structure, and let S be an upper subset of L, and let x be an element of L. If $x \in S$, then $\Uparrow x \subseteq S$.

(43) Let L be a non empty complete continuous Scott TopLattice, and let p be an element of L, and let S be a subset of L. If S is open and $p \in S$, then there exists an element q of L such that $q \ll p$ and $q \in S$.

(44) Let L be a non empty complete continuous Scott TopLattice and let p be an element of L. Then $\{\Uparrow q : q$ ranges over elements of L, $q \ll p\}$ is a basis of p.

(45) For every complete continuous Scott non empty TopLattice T holds $\{\Uparrow x : x$ ranges over elements of $T\}$ is a basis of T.

(46) Let T be a complete continuous Scott non empty TopLattice and let S be an upper subset of T. Then S is open if and only if S is Open.

(47) For every complete continuous Scott non empty TopLattice T and for every element p of T holds $\mathrm{Int}{\uparrow}p = {\Uparrow}p$.

(48) Let T be a complete continuous Scott non empty TopLattice and let S be a subset of T. Then $\mathrm{Int}\, S = \bigcup\{{\Uparrow}x : x$ ranges over elements of T, ${\Uparrow}x \subseteq S\}$.

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Received January 29, 1997

Automated Hyper-Linking in an Electronic Mathematical Proof-Checked Journal

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Table of Contents:

Principal Investigator.

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- **PI URL Home Page:** http:// math.uw.bialystok.pl/~trybulec
- **Grant Title:** Automated Hyper-Linking in an Electronic Mathematical Proof-Checked Journal
- **Grant/Contract Number:** N00014-95-1-1336
- **Mipr Number:** PL/FUW-BW71, NATO/CRG-951368
- **R&T Number:** 1115040rfw01, 1115040rfw02
- **Period of Performance:** 31 Aug 95 - 30 Aug 96
- **Today's Date:** 6 February 1997

Productivity Measures.

- **Number of refereed papers submitted not yet published:** 0
- **Number of refereed papers published:** 0

- **Number of refereed papers published:** 0
- **Number of unrefereed reports and articles:** 22
- **Number of books or parts thereof submitted but not published:** 0
- **Number of books or parts thereof published:** 0
- **Number of project presentations:** 0
- **Number of patents filed but not yet granted:** 0
- **Number of patents granted and software copyrights:** 0
- **Number of graduate students supported >= 25% of full time:** 4
- **Number of post-docs supported >= 25% of full time:** 0
- **Number of minorities supported:** 0

Summary of Objectives and Approach.

1. The main goal is to prove that in the case of an automatically generated mathematical text with sources which can be processed by the computer on the semantic level it is possible to build up automatically a rich system of hyperlinks. The objective is to find general rules of automatic hyper-linking for a specific set of documents: the articles published in the *Journal of Formalized Mathematics*.
2. Another goal is to translate to Mizar *A Compendium of Continuous Lattices* (by G.Gierz, K.H. Hoffman, K. Keimel, J.D Lawson, M. Mislove, and D.S. Scott - Springer-Verlag, Berlin, Heidelberg, New York 1980). We believe translating a part of a book is a better test for feasibility of practical formalization of mathematics than formalization of separate proofs. The concentration of efforts on a specific field of mathematics moves us closer to one of milestones of QED Project - to reach research frontiers in a field of mathematics.

Detailed Summary of Technical Progress.

1. The translation of whole MML (Mizar Mathematical Library) to JFM (Journal of Formalized Mathematics) format has been done in summer 95. This experiment resulted in debugging and enhancing the software used.
2. A preliminary design of JFM (Journal of Formalized Mathematics) has been completed (HTML-pages, directories, links).
3. In April 96 JFM was regenerated using corrected software. About 450 articles are published in JFM. The MML evolved fast between August 95 and April 96.
4. In August and September 95 a course on continuous lattices had been delivered (10 hours/week). Since November 95 it was continued as a seminar (2 hours/week).
5. The home pages of articles published in JFM have been redesigned. The references are moved to HTML page and the sections are linked to sections in postscript. However, the complete postscript versions of the publications are still kept (link: "download postscript version ").
6. The Journal has been moved to new host: http://mizar.uw.bialystok.pl/JFM/.
7. The system for inserting links (without human assistance) has been developed and it is used for linking. It inserted 288,578 links in 469 Mizar abstracts, about 600 links per article. New articles are linked when submitted and put to Preprints.
8. The articles written to formalize the Compendium are divided into two series **WAYBEL** and **YELLOW**.
   An article belonging to the **WAYBEL** series is a translation of part of the Compendium. Articles belonging to **YELLOW** series provide a bridge to Mizar Mathematical Library and cover the mathematical knowledge needs to write the **WAYBEL** articles but not formalized in MML (e.g Moore-Smith convergence).
   The most important observation is that MML was almost ready for formalization of the

The most important observation is that MML was almost ready for formalization of the Compendium. (The size of the **WAYBEL** series and the **YELLOW** series is of the same order: **WAYBEL** - 1,218,391 bytes, **YELLOW** - 604,813 bytes and the **WAYBEL** series grows faster).

9. The translation of the Compendium has reached page 105.

Transitions and DOD Interactions.

Software and Hardware Prototypes.

List of Publications.

1. Grzegorz Bancerek, *Bounds in Posets and Relational Substructures*, J.Form.Math., vol. 8.
2. Grzegorz Bancerek, *Directed Sets, Nets, Ideals, Filters, and Maps*, J.Form.Math., vol. 8.
3. Adam Grabowski and Robert Milewski, *Boolean Posets, Posets under Inclusion and Products of Relational Structures*, J.Form.Math., vol. 8.
4. Mariusz Zynel and Czeslaw Bylinski, *Properties of Relational Structures, Posets, Lattices and Maps*, J.Form.Math., vol. 8.
5. Czeslaw Bylinski, *Galois Connections*, J.Form.Math., vol. 8.
6. Artur Kornilowicz, *Cartesian Products of Relations and Relational Structures*, J.Form.Math., vol. 8.
7. Artur Kornilowicz, *Definitions and Properties of the Join and Meet of Subsets*, J.Form.Math., vol. 8.
8. Artur Kornilowicz, *Meet - continuous Lattices*, J.Form.Math., vol. 8.
9. Grzegorz Bancerek, *The "Way-Below" Relation*, J.Form.Math., vol. 8.
10. Adam Grabowski, *Auxiliary and Approximating Relations*, J.Form.Math., vol. 8.
11. Mariusz Zynel, *The Equational Characterization*, J.Form.Math., vol. 8.
12. Agnieszka Julia Marasik, *Miscellaneous Facts about Relation Structure*, J.Form.Math., vol. 8.
13. Andrzej Trybulec, *Moore-Smith Convergence*, J.Form.Math., vol. 8.
14. Grzegorz Bancerek, *Duality in Relation Structures*, J.Form.Math., vol. 8.
15. Beata Madras, *Irreducible and Prime Elements*, J.Form.Math., vol. 8.
16. Grzegorz Bancerek, *Prime Ideals and Filters*, J.Form.Math., vol. 8.
17. Robert Milewski, *Algebraic Lattices*, J.Form.Math., vol. 8.
18. Artur Kornilowicz, *On The Topological Properties of Meet-Continuous Lattices*, J.Form.Math., vol. 8.
19. Andrzej Trybulec, *Baire Spaces, Sober Spaces*, J.Form.Math., vol. 9.
20. Grzegorz Bancerek, *Closure Operators and Subalgebras*, J.Form.Math., vol. 9.
21. Andrzej Trybulec, *Scott Topology*, J.Form.Math., vol. 9.
22. Artur Kornilowicz, *On the Baire Category Theorem*, J.Form.Math., vol. 9.

Invited and Contributed Presentations.

March 16 - 19, 1996 TMR-NET Workshop, Rome, Italy
August 27-30, 1996 TPHOLs'96, Turku, Finland
November 18-20, 1996 TMR-NET Workshop, Dagstuhl, Germany

Honors, Prizes or Awards Received.

Project Personnel Promotions.

Project Staff.

1. **Name:** Dr Andrzej Trybulec
   - Homepage
   - **Position:** Associate Professor
   - **Task:** principal investigator
2. **Name:** Mr Grzegorz Bancerek
   - Homepage
   - **Position:** Assistant Professor
   - **Task:** developing automatically typeset electronic journal
3. **Name:** Mr Czeslaw Bylinski
   - Homepage
   - **Position:** Senior research scientist
   - **Task:** maintaining Journal of Formalized Mathematics
4. **Name:** Mr Adam Grabowski
   - Homepage
   - **Position:** assistant professor
   - **Task:** developing electronic mathematical journal
5. **Name:** Mr Artur Kornilowicz
   - Homepage
   - **Position:** assistant professor
   - **Task:** developing electronic mathematical journal
6. **Name:** Ms Beata Madras
   - Homepage
   - **Position:** Assistant Professor
   - **Task:** developing Mizar Mathematical Library
7. **Name:** Mr Roman Matuszewski
   - Homepage
   - **Position:** Senior research scientist
   - **Task:** co-principal investigator
8. **Name:** Mr Mariusz Zynel
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   - **Position:** assistant professor
   - **Task:** developing electronic mathematical journal

Multimedia URL.

Keywords.

1. Computer Oriented Formalization of Mathematics

1. Computer Oriented Formalization of Mathematics
2. Mathematical Proof-Checked Journal
3. Mizar Mathematical Library
4. Encoding Mathematics

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- **Business Office Fax Number:** +48 (85) 457-073
- **Business Office Email:** trybulec@math.uw.bialystok.pl

Expenditures

1. **Est. FY97:** 100 %
2. **FY96:** 100 %
3. **FY95:** 100 %

Current and Former Students

1. **Name:** Mr Adam Grabowski
   - Homepage
   - **Position:** assistant professor
   - **Nationality:** Polish
   - **Available for Summer at DoD Lab:** Yes
   - **Task:** developing electronic mathematical journal
   - **Thesis:** Inverse limits of manysorted algebras. Examples of category structures.
   - **Graduated:** 1996 Master of Mathematics
   - **Job:** Institute of Mathematics, Warsaw University, Bialystok Branch
2. **Name:** Mr Artur Kornilowicz
   - Homepage
   - **Position:** assistant professor
   - **Nationality:** Polish
   - **Available for Summer at DoD Lab:** Yes
   - **Task:** developing electronic mathematical journal
   - **Thesis:** Varieties of many sorted algebras
   - **Graduated:** 1996 Master of Mathematics
   - **Job:** Institute of Mathematics, Warsaw University, Bialystok Branch
3. **Name:** Ms Agnieszka Marasik
   - Homepage
   - **Position:** graduate
   - **Nationality:** Polish
   - **Available for Summer at DoD Lab:** No
   - **Task:** developing electronic mathematical journal
   - **Thesis:** Algebraic closure operators
   - **Job:**
4. **Name:** Mr Mariusz Zynel
   - Homepage

- **Position:** assistant professor
- **Nationality:** Polish
- **Available for Summer at DoD Lab:** Yes
- **Task:** developing electronic mathematical journal
- **Thesis:** Finite geometries over Grassmanians
- **Graduated:** 1996 Master of Mathematics
- **Job:** Institute of Mathematics, Warsaw University, Bialystok Branch

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Book Plans

Sabbatical Plans

Related Research

1. QED Project
2. Mizar Project
3. TMR-NET - Mathematical Assistant (Training and Mobility for Researchers in the European Union Programme)
4. OpenMath
5. IMPS Interactive Mathematical Proof System.
6. Formal Methods
7. HOL mechanical theorem proving system.
8. W3C HTML Math

History

1. Mizar MSE (sometimes called Baby Mizar) is a byproduct of the Mizar project used in teaching logic. More information in Mathesis Universalis - No.3 Mizar MSE
2. Mizar is distributed by anonymous ftp site in Warsaw . This site is mirrored in Canada, University of Alberta and on three sites in Japan: Shinshu University , Chiba University , Tokyo University .
3. Andrzej Trybulec visit in Argonne National Laboratory, May 1994, during QED Workshop I, supported by ONR (ONR Reference 5000-1133/93/A0068).
4. The QED Workshop II organized by our group, July 1995 in Warsaw, co-sponsored by ONR (ONR order N00014-95-M-0072).
5. John Harrison embedded a part of Mizar to HOL. He writes in abstract of *A Mizar Mode for HOL* in J. von Wright, J. Grudy, and J. Harrison *Theorem Proving in Higher Order Logics* in *Proceedings of TPHOLs'96*, pages 204-220. Springer, published 1996, LNCS 1125 "Existing HOL proofs styles are, however, very different from those used in textbooks. Here we describe the addition of another style, inspired by Mizar. We believe the resulting system combines the secure extensibility and interactivity of HOL with Mizar's readability and lack of logical prescriptiveness".

□